

PERIODIC CYCLIC HOMOLOGY AND EQUIVARIANT GERBES

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ABSTRACT. This paper is our first step in establishing a de Rham model for equivariant twisted K -theory using machinery from noncommutative geometry. Let G be a compact Lie group, M a compact manifold on which G acts smoothly. For any $\alpha \in H_G^3(M, \mathbb{Z})$ we introduce a notion of localized equivariant twisted cohomology $H^\bullet(\tilde{\Omega}^\bullet(M, G, L)_g, d_{Gg}^\alpha)$, indexed by $g \in G$. We prove that there exists a natural family of chain maps, indexed by $g \in G$, inducing a family of morphisms from the equivariant periodic cyclic homology $HP_\bullet^G(C^\infty(M, \alpha))$, where $C^\infty(M, \alpha)$ is a certain smooth algebra constructed from an equivariant bundle gerbe defined by $\alpha \in H_G^3(M, \mathbb{Z})$, to $H^\bullet(\tilde{\Omega}^\bullet(M, G, L)_g, d_{Gg}^\alpha)$. We formulate a conjecture of Atiyah-Hirzebruch type theorem for equivariant twisted K -theory.

1. INTRODUCTION

The well-known Atiyah-Hirzebruch theorem asserts that for a smooth manifold M , the Chern character establishes an isomorphism:

$$K^\bullet(M) \otimes \mathbb{C} \xrightarrow[\cong]{\text{ch}} H_{\text{DR}}^\bullet(M, \mathbb{C})$$

Therefore, modulo the torsion, K -theory groups are isomorphic to the (\mathbb{Z}_2 -graded) de Rham cohomology groups. In the study of K -theory, it has been a central question how to establish a Atiyah-Hirzebruch type theorem for other types of K -theory groups, among which are equivariant K -theory [21]. In 1994, Block-Getzler proved the following remarkable theorem [8] extending a result of Baum-Brylinski-MacPherson [3] in the case of $G = S^1$:

Let G be a compact Lie group, M a compact manifold on which G acts smoothly. Then

$$K_G^\bullet(M) \otimes_{R(G)} R^\infty(G) \xrightarrow[\cong]{} H^\bullet(\mathcal{A}_G^\bullet(M), d_{\text{eq}}).$$

Here $R(G)$ is the representation ring of G , and $R^\infty(G)$ is the ring of smooth functions on G invariant under the conjugation. Then $R^\infty(G)$ is an algebra over $R(G)$, since $R(G)$ maps to $R^\infty(G)$ by the character map. And $H^\bullet(\mathcal{A}_G^\bullet(M), d_{\text{eq}})$ is the cohomology of global equivariant differential forms on M , also called the *delocalized equivariant* cohomology. Roughly speaking, $H^\bullet(\mathcal{A}_G^\bullet(M), d_{\text{eq}})$ can be considered as the cohomology of the inertia stack $\Lambda[M/G]$ (while ordinary equivariant cohomology is the cohomology of the quotient stack $[M/G]$). In a certain sense, delocalized equivariant cohomology $H^\bullet(\mathcal{A}_G^\bullet(M), d_{\text{eq}})$ is a de Rham description of the equivariant K -theory.

In late 1980's, Block [7] and Brylinski [10, 11] independently proved the following theorem:

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Let G be a compact Lie group and A a topological G -algebra. Then the equivariant Chern character (26) induces an isomorphism

$$(1) \quad HP_{\bullet}^G(A) \xrightarrow[\cong]{} K_{\bullet}^G(A) \otimes_{R(G)} R^{\infty}(G).$$

By using the above theorem, Block-Getzler [8] reduces the problem of establishing Atiyah-Hirzebruch type theorem for equivariant K -theory to that of computing the equivariant periodic cyclic homology $HP_{\bullet}^G(C^{\infty}(M))$, and thus can apply the machinery of noncommutative geometry. When $G = \{*\}$, $HP_{\bullet}^G(C^{\infty}(M))$ is isomorphic to the (\mathbb{Z}_2 -graded) de Rham cohomology $H_{DR}^{\bullet}(M)$ according to a theorem of Connes [12, 13], and therefore Block-Getzler theorem reduces to the classical Atiyah-Hirzebruch theorem.

Motivated by string theory, there has been a great deal of interest in the study of twisted K -theory [9, 31]. It is thus natural to ask how to extend Atiyah-Hirzebruch theorem to twisted K -theory. The manifolds case was completely solved in [18], while orbifolds case was done [26]. This paper is our first step in establishing Atiyah-Hirzebruch type theorem for twisted equivariant K -theory.

In literatures, there exist different equivalent approaches to twisted equivariant K -theory, eg. [2]. In [25], with Laurent-Gengoux, we introduced twisted equivariant K -theory based on the idea from noncommutative geometry. It can be described roughly as follows. For a compact manifold M equipped with an action of a compact Lie group G , and any $\alpha \in H_G^3(M, \mathbb{Z})$, one can always construct an S^1 -central extension of Lie groupoids $\tilde{X}_1 \xrightarrow{\pi} X_1 \rightrightarrows X_0$ representing $\alpha \in H_G^3(M, \mathbb{Z})$. Such an S^1 -central extension is unique up to Morita equivalence. From the S^1 -central extension of Lie groupoids $\tilde{X}_1 \xrightarrow{\pi} X_1 \rightrightarrows X_0$, one constructs a convolution algebra $C_c(X_1, L)$. Then the twisted equivariant K -theory groups can be defined as the K -theory groups of this algebra (or its corresponding reduced C^* -algebra), i.e., $K_{G,\alpha}^{\bullet}(M) := K_{\bullet}(C_c(X_1, L))$.

In order to apply Block-Brylinski theorem [7, 10, 11] (Theorem 5.7) in our situation, first of all, we prove the following

Theorem A. *For any integer class $\alpha \in H_G^3(M, \mathbb{Z})$, there always exists a G -equivariant bundle gerbe [27] $\tilde{H}_1 \xrightarrow{\phi} H_1 \rightrightarrows H_0$ over M with an equivariant connection and an equivariant curving, whose equivariant 3-curvature represents α in the Cartan model $(\Omega_G^{\bullet}(M), d + \iota)$.*

As a consequence, equivalently one can define the twisted equivariant K -theory groups $K_{G,\alpha}^{\bullet}(M)$ as $K_{\bullet}^G(C_c^{\infty}(H, L))$, where $C_c^{\infty}(H, L)$ is the convolution algebra corresponding to the G -equivariant bundle gerbe $\tilde{H}_1 \xrightarrow{\phi} H_1 \rightrightarrows H_0$. Note that when $G = \{*\}$ and $\alpha \in H_G^3(M, \mathbb{Z})$ being trivial, $\tilde{H}_1 \xrightarrow{\phi} H_1 \rightrightarrows H_0$ is Morita equivalent to $C^{\infty}(M)$.

Therefore, we are led to the following

Problem B. *Compute the equivariant periodic cyclic homology $HP_{\bullet}^G(C_c^{\infty}(H, L))$ in terms of geometric data to obtain de Rham type cohomology groups.*

Toward this direction, we first prove the following:

Theorem C. *Any Lie groupoid S^1 -central extension representing $\alpha \in H_G^3(M, \mathbb{Z})$ canonically induces a family of G -equivariant flat S^1 -bundles $\coprod_{g \in G} (P^g \rightarrow M^g)$ indexed by $g \in G$, where $M^g = \{x \in M | x \cdot g = x\}$ is the fixed point set under the diffeomorphism*

$x \rightarrow x \cdot g$, and for any $h \in G$, the action by h , is an isomorphism of the flat bundles $(P^g \rightarrow M^g) \mapsto (P^{h^{-1}gh} \rightarrow M^{h^{-1}gh})$.

Two such S^1 -central extensions induce G -equivariantly isomorphic families of flat S^1 -bundles.

Let $L = \coprod_{g \in G} L^g$, where $L^g = P^g \times_{S^1} \mathbb{C}$, $\forall g \in G$, are their associated G -equivariant flat complex line bundles.

Choose a G -equivariant closed 3-form $\eta_G \in Z_G^3(M)$ such that $[\eta_G] = \alpha$. Following Block-Getzler [8], we consider the localized twisted equivariant cohomology as follows. Denote by $\bar{\Omega}^\bullet(M, G, L)_g$ the space $\bar{\Omega}_{G^g}(M^g, L^g)$ of germs at zero of G^g -equivariant smooth maps from \mathfrak{g}^g to $\Omega^\bullet(M^g, L^g)$. By $d_{G^g}^\alpha$ we denote the twisted equivariant differential operator $\nabla_g + \iota - 2\pi i \eta_G$ on $\bar{\Omega}_{G^g}(M^g, L^g)$, where $\nabla_g : \bar{\Omega}_{G^g}(M^g, L^g) \rightarrow \bar{\Omega}_{G^g}(M^g, L^g)$ is the covariant differential induced by the flat connection on the complex line bundle $L^g \rightarrow M^g$, and η_G acts on $\bar{\Omega}_{G^g}(M^g, L^g)$ by taking the wedge product with $i_g^* \eta_G$. Here $i_g^* : \Omega_G(M) \rightarrow \Omega_{G^g}(M^g)$ is the restriction map. It is simple to see that $(d_{G^g}^\alpha)^2 = 0$, and $\{d_{G^g}^\alpha | g \in G\}$ are compatible with the G -action. The family of cohomology groups are denoted by $H^\bullet(\bar{\Omega}^\bullet(M, G, L)_g, d_{G^g}^\alpha)$, and are called *localized twisted equivariant cohomology*.

Our main result in the paper is the following:

Theorem D. *There exists a family of G -equivariant chain maps, indexed by $g \in G$:*

$$\tau_g : (PC_\bullet^G(C_c^\infty(H, L)), b + \mathcal{B}) \rightarrow (\bar{\Omega}^\bullet(M, G, L)_g, d_{G^g}^\alpha)$$

By G -equivariant chain maps, we mean that the following diagram of chain maps commutes:

$$\begin{array}{ccc} (PC_\bullet^G(C_c^\infty(H, L)), b + \mathcal{B}) & \xrightarrow{\tau_g} & ((\bar{\Omega}^\bullet(M, G, L)_g, d_{G^g}^\alpha)) \\ & \searrow \tau_{h^{-1}gh} & \downarrow \phi_h \\ & & ((\bar{\Omega}^\bullet(M, G, L)_{h^{-1}gh}, d_{G^{h^{-1}gh}}^\alpha)). \end{array}$$

Therefore, there is a family of morphisms on the level of cohomology:

$$(2) \quad \tau_g : HP_\bullet^G(C_c^\infty(H, L)) \rightarrow H^\bullet((\bar{\Omega}^\bullet(M, G, L)_g, d_{G^g}^\alpha)).$$

In order to completely solve Problem B, following Block-Getzler [8], we propose the following

Problem E. *Introduce global twisted equivariant differential forms by modifying the notion of global equivariant differential forms a la Block-Getzler [8] to define delocalized twisted equivariant cohomology $H_{G, \text{delocalized}, \alpha}^\bullet(M)$, and establish the isomorphism*

$$HP_\bullet^G(C_c^\infty(H, L)) \xrightarrow{\cong} H_{G, \text{delocalized}, \alpha}^\bullet(M).$$

We will devote Section 5 to discussions on this issue.

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2. LOCALIZED EQUIVARIANT TWISTED COHOMOLOGY

2.1. **S^1 -gerbes over $[M/G]$.** Let $\tilde{X}_1 \rightarrow X_1 \rightrightarrows X_0$ be an S^1 -central extension of Lie groupoids. By abuse of notations, we denote, by d , both the de Rham differentials $\Omega^\bullet(X_\bullet) \rightarrow \Omega^{\bullet+1}(X_\bullet)$, and $\Omega^\bullet(\tilde{X}_\bullet) \rightarrow \Omega^{\bullet+1}(\tilde{X}_\bullet)$. And, by ∂ , we denote the simplicial differential $\partial : \Omega^\bullet(X_\bullet) \rightarrow \Omega^\bullet(X_{\bullet+1})$ for the groupoid $X_1 \rightrightarrows X_0$, while, by $\tilde{\partial}$, we denote the simplicial differential $\tilde{\partial} : \Omega^\bullet(\tilde{X}_\bullet) \rightarrow \Omega^\bullet(\tilde{X}_{\bullet+1})$ for the groupoid $\tilde{X}_1 \rightrightarrows X_0$. See Section 2.1 [32] for details on de Rham cohomology of Lie groupoids. Recall the following

Definition 2.1 ([5]). (i) A connection form $\theta \in \Omega^1(\tilde{X}_1)$ for the S^1 -bundle $\tilde{X}_1 \rightarrow X_1$, such that $\tilde{\partial}\theta = 0$, is a *connection* on the S^1 -central extension $S^1 \rightarrow \tilde{X}_1 \rightarrow X_1 \rightrightarrows X_0$;
(ii) Given θ , a 2-form $B \in \Omega^2(X_0)$, such that $d\theta = \tilde{\partial}B$ is a *curving*;
(iii) and given (θ, B) , the 3-form $\Omega = dB \in H^0(X_\bullet, \Omega^3) \subset \Omega^3(X_0)$ is called the *3-curvature*.

We have the following

Lemma 2.2. *Given an S^1 -central extension $\tilde{X}_1 \xrightarrow{\pi} X_1 \rightrightarrows X_0$,*

- (1) *the obstruction group to the existence of connections is $H^2(X_\bullet, \Omega^1)$;*
- (2) *the obstruction group to the existence of curvings is $H^1(X_\bullet, \Omega^2)$.*

PROOF. (1). Choose any connection one-form $\theta \in \Omega^1(\tilde{X}_1)$ of the S^1 -bundle $\pi : \tilde{X}_1 \rightarrow X_1$. It is simple to see that $\tilde{\partial}\theta = \pi^*\eta$, where $\eta \in \Omega^1(X_2)$. Here by abuse of notations, we use the same symbol π to denote the induced projection $\tilde{X}_2 \rightarrow X_2$. Since $\pi^*\partial\eta = \tilde{\partial}\pi^*\eta = \tilde{\partial}^2\theta = 0$, it thus follows that $\partial\eta = 0$. If θ' is another connection one-form, then θ' differs from θ by a one-form $A \in \Omega^1(X_1)$. Thus $\eta' = \eta + \partial A$. It follows that the class $[\eta] \in H^2(\Omega^1(X_\bullet), \partial)$ is independent of the choice of θ . Note that $H^\bullet(\Omega^1(X_\bullet), \partial) \cong H^\bullet(X_\bullet, \Omega^1)$ since Ω^1 is a soft sheaf.

Assume that $[\eta] \in H^2(X_\bullet, \Omega^1)$ vanishes. We may write $\eta = \partial\alpha$ for some one-form $\alpha \in \Omega^1(X_1)$. It is simple to see that $\theta' = \theta - \pi^*\alpha$ is indeed a connection for the S^1 -extension.

(2). Assume that $\theta \in \Omega^1(\tilde{X}_1)$ is a connection. Let $\omega \in \Omega^2(X_1)$ be its curvature, i.e. $d\theta = \pi^*\omega$. Since $\pi^*\partial\omega = \tilde{\partial}\pi^*\omega = \tilde{\partial}d\theta = d\tilde{\partial}\theta = 0$, we have $\partial\omega = 0$. Hence $[\omega] \in H^1(\Omega^2(X_\bullet), \partial) \cong H^1(X_\bullet, \Omega^2)$. Then $[\omega]$ vanishes if and only if there exists $B \in \Omega^2(X_0)$ such that $\omega = \partial B$, i.e. $d\theta = \tilde{\partial}B$. Hence $[\omega] = 0$ if and only if there exists a curving. \square

Remark 2.3. Note that $H^2(X_\bullet, \Omega^1)$ and $H^1(X_\bullet, \Omega^2)$ are isomorphic to $H^2(\mathfrak{X}, \Omega^1)$ and $H^1(\mathfrak{X}, \Omega^2)$, respectively, where \mathfrak{X} is the differentiable stack corresponding to the groupoid $X_1 \rightrightarrows X_0$ [5]. Therefore, these cohomology groups are Morita invariant.

The following theorem is due to Abad-Crainic [[1] Corollary 4.2].

Theorem 2.4. *If $X_1 \rightrightarrows X_0$ is a proper Lie groupoid, then*

$$H^p(X_\bullet, \Omega^q) = 0, \quad \text{if } p > q.$$

In particular, we have $H^2(X_\bullet, \Omega^1) = 0$. Thus we have the following

Proposition 2.5. *If $\tilde{X}_1 \rightarrow X_1 \rightrightarrows X_0$ is an S^1 -central extension of a proper Lie groupoid $X_1 \rightrightarrows X_0$, then connections always exist.*

In particular, if G is a compact Lie group acting on a manifold M , and $\tilde{X}_1 \xrightarrow{\pi} X_1 \rightrightarrows X_0$ is an S^1 -central extension representing any class $\alpha \in H_G^3(M, \mathbb{Z})$, then this extension admits a connection.

Note that $H^1((M \rtimes G)_\bullet, \Omega^2)$ may not necessarily vanish in general, so curvings does not always exist.

2.2. Equivariant bundle gerbes. We now recall some basic notions regarding equivariant cohomology in order to fix the notations.

Let M be a smooth manifold with a smooth right action of a compact Lie group G : $(x, g) \in M \times G \rightarrow x \cdot g$. There is an induced action of the group G on the space $\Omega^\bullet(M)$ of differential forms on M by $g \cdot \omega = R_g^* \omega$, where $R_g : M \rightarrow M$ is the operation of the action by $g \in G$. If $\omega : \mathfrak{g} \rightarrow \Omega^\bullet(M)$ is a map from \mathfrak{g} to $\Omega^\bullet(M)$, the group G acts on ω by the formula

$$(3) \quad (g \cdot \omega)(X) = g \cdot (\omega(\text{Ad}_{g^{-1}} X)), \quad \forall X \in \mathfrak{g}.$$

By an equivariant differential form on M , we mean a G -equivariant polynomial function $\omega : \mathfrak{g} \rightarrow \Omega^\bullet(M)$. When ω is a homogeneous polynomial, the degree of ω is defined to be the sum of $2 \times$ the degree of the polynomial and the degree of the differential form $\omega(X)$, $X \in \mathfrak{g}$. We denote by $\Omega_G^k(M)$ the space of local equivariant differential forms of degree k . Define an equivariant differential $d_G = d + \iota : \Omega_G^\bullet(M) \rightarrow \Omega_G^{\bullet+1}(M)$, where

$$(d\omega)(X) = d(\omega(X)), \quad (\iota\omega)(X) = \iota_{\hat{X}}\omega(X).$$

Here \hat{X} denotes the infinitesimal vector field on M generated by the Lie algebra element $X \in \mathfrak{g}$. Note that $\Omega_G^\bullet(M)$ can also be identified with the space of invariant polynomials $(S\mathfrak{g}^* \otimes \Omega^\bullet(M))^G$. The cochain complex $(\Omega_G^\bullet(M), d_G)$ is called the Cartan model of the equivariant cohomology group $H_G^\bullet(M)$ [6].

Following [27], an S^1 -central extension $\tilde{H}_1 \xrightarrow{p} H_1 \rightrightarrows H_0$ is said to be G -equivariant if both $\tilde{H}_1 \rightrightarrows H_0$ and $H_1 \rightrightarrows H_0$ are G -groupoids, the groupoid morphism $p : \tilde{H}_1 \rightarrow H_1$ is G -equivariant, and $\tilde{H}_1 \xrightarrow{p} H_1$ is a G -equivariant principal S^1 -bundle, i.e. if the following relations:

$$\begin{aligned} (\tilde{x} \cdot \tilde{y}) \star g &= (\tilde{x} \star g) \cdot (\tilde{y} \star g) \\ p(\tilde{x} \star g) &= p(\tilde{x}) \star g \\ (\lambda \tilde{x}) \star g &= \lambda(\tilde{x} \star g) \end{aligned}$$

are satisfied for all $g \in G$, all composable pairs (\tilde{x}, \tilde{y}) in \tilde{H}_2 and all $\lambda \in S^1$.

Now assume that $H_0 \xrightarrow{\pi} M$ is a G -equivariant surjective submersion. Consider the pair groupoid $H_1 \rightrightarrows H_0$, where $H_1 = H_0 \times_M H_0$, the source and target maps are $t(x, y) = x$ and $s(x, y) = y$, and the multiplication $(x, y) \cdot (y, z) = (x, z)$. Then $H_1 \rightrightarrows H_0$ is a G -groupoid, which is Morita equivalent to the G -manifold $M \rightrightarrows M$. A G -equivariant bundle gerbe [19, 27] is a G -equivariant S^1 -central extension of Lie groupoids $\tilde{H}_1 \xrightarrow{p} H_1 \rightrightarrows H_0$.

The following notion is due to Stienon [27].

Definition 2.6 ([27]). (1) An equivariant connection is a G -invariant 1-form $\theta \in \Omega^1(\tilde{H}_1)^G$ such that θ is a connection 1-form for the principal S^1 -bundle $\tilde{H}_1 \xrightarrow{p} H_1$ and satisfies

$$\tilde{\partial}\theta = 0.$$

(2) Given an equivariant connection θ , an equivariant curving is a degree-2 element $B_G \in \Omega_G^2(H_0)$ such that

$$(4) \quad \text{curv}_G(\theta) = \partial B_G,$$

where $\text{curv}_G(\theta)$ denotes the equivariant curvature of the S^1 -principal bundle $\tilde{H}_1 \xrightarrow{p} H_1$, i.e. the element $\text{curv}_G(\theta) \in \Omega_G^2(H_1)$ characterized by the relation

$$(5) \quad d_G \theta = p^* \text{curv}_G(\theta).$$

- (3) Given an equivariant connection and an equivariant curving (θ, B_G) , the corresponding equivariant 3-curvature is the equivariant 3-form $\eta_G \in \Omega_G^3(M)$ such that

$$(6) \quad \pi^* \eta_G = d_G B_G.$$

Here the coboundary operators $\partial : \Omega^\bullet(H_\bullet) \rightarrow \Omega^\bullet(H_{\bullet+1})$ and $\tilde{\partial} : \Omega^\bullet(\tilde{H}_\bullet) \rightarrow \Omega^\bullet(\tilde{H}_{\bullet+1})$ are the simplicial differentials of the groupoids $H_1 \rightrightarrows H_0$ and $\tilde{H}_1 \rightrightarrows H_0$, respectively.

The following result can be found in [27].

Proposition 2.7. *Let $\tilde{H}_1 \xrightarrow{\phi} H_1 \rightrightarrows H_0$ be a G -equivariant bundle gerbe over a G -manifold M .*

- (1) *Equivariant connections and equivariant curvings (θ, B_G) always exist.*
- (2) *The class $[\eta_G] \in H_G^3(M)$ defined by the equivariant 3-curvature is independent of the choice of θ and B_G .*

The degree 3 equivariant cohomology class $[\eta_G] \in H_G^3(M)$ is called the *equivariant Dixmier-Douady class* of the underlying equivariant bundle gerbe.

2.3. Kostant-Weil theorem for equivariant bundle gerbes. The main result of this section is the following Kostant-Weil type of quantization theorem for equivariant bundle gerbes.

Theorem 2.8. *For any integer class $\alpha \in H_G^3(M, \mathbb{Z})$, there always exists a G -equivariant bundle gerbe over M with an equivariant connection and an equivariant curving, whose equivariant 3-curvature represents α in the Cartan model. Moreover, one can choose a G -basic connection as an equivariant connection on the G -equivariant bundle gerbe.*

The rest of this subsection is devoted to the proof of Theorem 2.8.

For any $\alpha \in H_G^3(M, \mathbb{Z}) \cong H^3((M \rtimes G)_\bullet, \mathbb{Z})$, let $\tilde{\Gamma} \xrightarrow{\pi} \Gamma \rightrightarrows M'$, where $p : M' \rightarrow M$ is an immersion, be an S^1 -central extension of Lie groupoids representing α . Let $H_0 = M' \times G$. Assume that the map

$$\pi : H_0 \rightarrow M, \quad \pi(x, g) = p(x)g$$

is a surjective submersion. Note that this assumption always holds if $M' = \coprod U_i$ is an open cover of M . It is clear that H_0 admits a right G -action: $(x, h) \cdot g = (x, hg)$, $\forall (x, h) \in M' \times G$ and $g \in G$. Then the map $\pi : H_0 \rightarrow M$ is clearly G -equivariant. Let $H_1 = H_0 \times_M H_0$. Then $H_1 \rightrightarrows H_0$ is a G -groupoid, which is Morita equivalent to $M \rightrightarrows M$. Consider the groupoid morphism

$$\nu : H_1 \rightarrow \Gamma, \quad \nu((x, g), (y, h)) = (x, gh^{-1}, y).$$

Let $\tilde{H}_1 \rightarrow H_1$ be the pullback S^1 -bundle of $\tilde{\Gamma} \rightarrow \Gamma$ via the map $\nu : H_1 \rightarrow \Gamma$:

$$\tilde{H}_1 = H_1 \times_\Gamma \tilde{\Gamma} = \{((x, g), (y, h), \tilde{r}) | (x, g), (y, h) \in H_0, \tilde{r} \in \tilde{\Gamma}, p(x)g = p(y)h, (x, gh^{-1}, y) = p(\tilde{r})\}. \blacksquare$$

Since ν is constant along the G -orbits on H_1 , the G -action on H_1 naturally lifts to \tilde{H}_1 , i.e.

$$((x, g), (y, h), \tilde{r}) \cdot k = ((x, gk), (y, hk), \tilde{r}), \quad \forall k \in G, ((x, g), (y, h), \tilde{r}) \in \tilde{H}_1.$$

Thus $\tilde{H}_1 \rightrightarrows H_0$ is a G -equivariant groupoid. Let $\phi : \tilde{H}_1 \rightarrow H_1$ be the projection.

Lemma 2.9. (1) $\tilde{H}_1 \xrightarrow{\phi} H_1 \rightrightarrows H_0$ is a G -equivariant bundle gerbe over M ;

(2) the following diagram

$$(7) \quad \begin{array}{ccc} \tilde{H}_1 & \xrightarrow{\tilde{q}} & \tilde{\Gamma} \\ \downarrow & & \downarrow \\ H_1 & \xrightarrow{q} & \Gamma \\ \Downarrow & & \Downarrow \\ H_0 & \xrightarrow{q_0} & M' \end{array}$$

where $\tilde{q} : \tilde{H}_1 \rightarrow \tilde{\Gamma}$, $((x, g), (y, h), \tilde{r}) \mapsto \tilde{r}$ is the projection, $q : H_1 \rightarrow \Gamma$ is the map ν , and $q_0 : H_0 (\cong M' \times G) \rightarrow M'$ is the projection pr_1 , defines a morphism of S^1 -central extensions;

(3) the S^1 -central extensions $\tilde{H}_1 \rtimes G \xrightarrow{\phi} H_1 \rtimes G \rightrightarrows H_0$ and $\tilde{\Gamma} \xrightarrow{\pi} \Gamma \rightrightarrows M'$ are Morita equivalent. Indeed,

$$(8) \quad \begin{array}{ccc} \tilde{H}_1 \rtimes G & \xrightarrow{\tilde{q} \circ pr_1} & \tilde{\Gamma} \\ \downarrow & & \downarrow \\ H_1 \rtimes G & \xrightarrow{q \circ pr_1} & \Gamma \\ \Downarrow & & \Downarrow \\ H_0 & \xrightarrow{q_0} & M' \end{array}$$

is a Morita morphism of S^1 -central extensions [25].

PROOF. This can be verified directly, which is left to the reader. \square

As an immediate consequence, we have the following

Corollary 2.10. *For any integer class $\alpha \in H_G^3(M, \mathbb{Z})$, there exists a G -equivariant bundle gerbe $\tilde{H}_1 \rightarrow H_1 \rightrightarrows H_0$ over M such that the Dixmier-Douady class of $\tilde{H}_1 \rtimes G \xrightarrow{\phi} H_1 \rtimes G \rightrightarrows H_0$ is equal to α under the natural isomorphism*

$$H^3((H \rtimes G)_\bullet, \mathbb{Z}) \cong H^3((M \rtimes G)_\bullet, \mathbb{Z}) \cong H_G^3(M, \mathbb{Z}).$$

PROOF. For any $\alpha \in H_G^3(M, \mathbb{Z}) \cong H^3((M \rtimes G)_\bullet, \mathbb{Z})$, according to Proposition 4.16 in [5], one can always represent α by an S^1 -central extension [5, 24] $\tilde{\Gamma} \xrightarrow{\pi} \Gamma \rightrightarrows M'$, where $p : M' \rightarrow M$ is étale and surjective, and $\Gamma = \{(x, g, y) \in M' \times G \times M' \mid p(x)g = p(y)\}$ is the pull-back groupoid of $M \rtimes G \rightrightarrows M$ via p . Then

$$(9) \quad \begin{array}{ccc} \Gamma & \xrightarrow{p} & M \rtimes G \\ \Downarrow & & \Downarrow \\ M' & \xrightarrow{p} & M \end{array}$$

is a Morita morphism, where, by abuse of notations, $p : \Gamma \rightarrow M \rtimes G$ is given by $p(x, g, y) = (p(x), g)$. The groupoid structure on $\Gamma \rightrightarrows M'$ is given by $s(x, g, y) = y$, $t(x, g, y) = x$, $(x, g, y)(y, h, z) = (x, gh, z)$ and $(x, g, y)^{-1} = (y, g^{-1}, x)$. Thus we are in the situation described at the beginning of this section. In particular, the Dixmier-Douady class of the S^1 -central extension $\tilde{H}_1 \rtimes G \rightarrow H_1 \rtimes G \rightrightarrows H_0$ is equal to α under the isomorphism $H^3((H \rtimes G)_\bullet, \mathbb{Z}) \cong H^3((M \rtimes G)_\bullet, \mathbb{Z}) = H_G^3(M, \mathbb{Z})$. This concludes the proof of the corollary. \square

Proof of Theorem 2.8 It remains to prove that the Dixmier-Douady class, in the Cartan model, defined by the equivariant 3-curvature of the G -equivariant bundle gerbe $\tilde{H}_1 \rightarrow H_1 \rightrightarrows H_0$ in Corollary 2.10 is equal to α . This essentially follows from a theorem of Stienon (Theorem 4.6 in [27]). More specifically, if $\eta_G \in Z_G^3(M)$ is the equivariant 3-curvature of the G -equivariant bundle gerbe $\tilde{H}_1 \rightarrow H_1 \rightrightarrows H_0$, then $[\eta_G]$ maps to the Dixmier-Douady class of the central extension $\tilde{H}_1 \rtimes G \rightarrow H_1 \rtimes G \rightrightarrows H_0$, under the isomorphism $H_G^3(M) \xrightarrow{\sim} H_{\text{DR}}^3((H \rtimes G)_\bullet)$ according to Theorem 4.6 in [27]. The latter corresponds exactly to α according to Corollary 2.10.

Finally, let $\theta' \in \Omega^1(\tilde{\Gamma})$ be a connection of the S^1 -central extension $\tilde{\Gamma} \xrightarrow{\pi} \Gamma \rightrightarrows M'$, which always exists according to Proposition 2.5. Let $\theta = \tilde{q}^* \theta' \in \Omega^1(\tilde{H}_1)$, where \tilde{q} is as in Eq. (7). Since \tilde{q} is a morphism of S^1 -central extensions, θ is clearly a connection of the S^1 -central extension $\tilde{H}_1 \xrightarrow{\phi} H_1 \rightrightarrows H_0$. Since $\tilde{q}(\tau \cdot g) = \tilde{q}(\tau)$, $\forall \tau \in \tilde{H}_1$, $g \in G$, it follows that θ is G -invariant. Since $\hat{X} \lrcorner \theta = 0$, $\forall X \in \mathfrak{g}$, thus θ is G -basic. This concludes the proof of the theorem.

Remark 2.11. When G is a compact simple Lie group and G acts on G by conjugation, there is an explicit construction of G -equivariant bundle gerbe due to Meinrenken [19] (see also [16] for the case of $G = SU(n)$).

2.4. Geometric transgression. For a Lie groupoid $\Gamma \rightrightarrows \Gamma_0$, by $S\Gamma$ we denote the space of closed loops $\{g \in \Gamma | s(g) = t(g)\}$. Then Γ acts on $S\Gamma$ by conjugation: $\gamma \cdot \tau = \gamma \tau \gamma^{-1}$, $\forall \gamma \in \Gamma$ and $\tau \in S\Gamma$ such that $s(\gamma) = t(\tau)$. One forms the transformation groupoid $\Lambda\Gamma : S\Gamma \rtimes \Gamma \rightrightarrows S\Gamma$, which is called the *inertia groupoid*. If $\tilde{\Gamma} \rightarrow \Gamma \rightrightarrows \Gamma_0$ is an S^1 -central extension, then the restriction $\tilde{\Gamma}|_{S\Gamma}$ is naturally endowed with an action of Γ . To see this, for any $\gamma \in \Gamma$, let $\tilde{\gamma} \in \tilde{\Gamma}$ be any of its lifting. Then for any $\tilde{\tau} \in \tilde{\Gamma}|_{S\Gamma}$ such that $s(\gamma) = t(\tilde{\tau})$, set

$$(10) \quad \tilde{\tau} \cdot \gamma = \tilde{\gamma}^{-1} \tilde{\tau} \tilde{\gamma}.$$

It is simple to see that this Γ -action is well defined, i.e. the right hand side of Eq. (10) is independent of the choice of the lifting $\tilde{\gamma}$. Thus $\tilde{\Gamma}|_{S\Gamma} \rightarrow S\Gamma$ naturally carries an S^1 -bundle structure over the inertia groupoid $\Lambda\Gamma \rightrightarrows S\Gamma$ (see also Proposition 2.9 in [28]).

Proposition 2.12. *Let $\Gamma \rightrightarrows \Gamma_0$ be a Lie groupoid. Then any S^1 -central extension $\tilde{\Gamma} \rightarrow \Gamma \rightrightarrows \Gamma_0$ induces an S^1 -bundle over the inertia groupoid $\Lambda\Gamma \rightrightarrows S\Gamma$.*

Remark 2.13. In general, the inertia groupoid is not a Lie groupoid since $S\Gamma$ may not be a smooth manifold.

Now assume that $\tilde{\Gamma} \xrightarrow{\pi} \Gamma \rightrightarrows \Gamma_0$ is an S^1 -central extension representing $\alpha \in H_G^3(M, \mathbb{Z})$. By Proposition 2.5, this central extension admits a connection $\theta \in \Omega^1(\tilde{\Gamma})$. According to Proposition 3.9 [28], there exists an induced connection on the associated S^1 -bundle $\tilde{\Gamma}|_{S\Gamma} \rightarrow S\Gamma$ over the inertia groupoid $\Lambda\Gamma$. Since $\Gamma \rightrightarrows \Gamma_0$ is Morita equivalent to $M \rtimes G \rightrightarrows M$, $\Lambda\Gamma$ is Morita equivalent to $\Lambda(M \rtimes G)$, where the Morita equivalence bimodule is induced by the equivalence bimodule between $\Gamma \rightrightarrows \Gamma_0$ and $M \rtimes G \rightrightarrows M$. Thus one obtains an S^1 -bundle P with a connection over the groupoid $\Lambda(M \rtimes G)$, according to Corollary 3.15 in [25].

It is clear that the groupoid $\Lambda(M \rtimes G)$ is isomorphic to $(\coprod_{g \in G} M^g) \rtimes G \rightrightarrows \coprod_{g \in G} M^g$, where $M^g = \{x \in M | x \cdot g = x\}$ is the fixed point set under the diffeomorphism $x \rightarrow x \cdot g$. By $\coprod_{g \in G} P^g \rightarrow \coprod_{g \in G} M^g$, we denote this S^1 -bundle. As a consequence, we obtain a family (i.e. over a manifold instead of over a groupoid) of S^1 -bundles $P^g \rightarrow M^g$, with connections, indexed by $g \in G$, on which G acts equivariantly preserving the connections. The main result of this section is the following

Theorem 2.14. *Let $\alpha \in H_G^3(M, \mathbb{Z})$.*

- (1) *Assume that $\tilde{\Gamma} \xrightarrow{\pi} \Gamma \rightrightarrows \Gamma_0$ is an S^1 -central extension representing α , where $\Gamma \rightrightarrows \Gamma_0$ is the pullback groupoid of the transformation groupoid $M \rtimes G \rightrightarrows M$ via a surjective submersion $p : \Gamma_0 \rightarrow M$. Then $\tilde{\Gamma} \xrightarrow{\pi} \Gamma \rightrightarrows \Gamma_0$ canonically induces a family of G -equivariant flat S^1 -bundles $P^g \rightarrow M^g$ indexed by $g \in G$.*

Here, for any $h \in G$, the action by h , denoted R_h by abuse of notations, is an isomorphism of the flat bundles $(P^g \rightarrow M^g) \mapsto (P^{h^{-1}gh} \rightarrow M^{h^{-1}gh})$ over the map $R_h : M^g \rightarrow M^{h^{-1}gh}$; Moreover, g acts on P^g as an identity.

- (2) *If $\tilde{\Gamma} \xrightarrow{\pi} \Gamma \rightrightarrows \Gamma_0$ and $\tilde{\Gamma}' \xrightarrow{\pi'} \Gamma' \rightrightarrows \Gamma'_0$ are any two such S^1 -central extensions, their induced families of flat S^1 -bundles $\coprod_{g \in G} (P^g \rightarrow M^g)$ and $\coprod_{g \in G} (P'^g \rightarrow M^g)$ are G -equivariantly isomorphic.*

We need a few lemmas. Note that $\Gamma \cong \{(x, g, y) \in \Gamma_0 \times G \times \Gamma_0 \mid p(x)g = p(y)\}$ and $S\Gamma = \{(x, g, x) \in \Gamma_0 \times G \times \Gamma_0 \mid p(x)g = p(x)\}$. For any $g \in G$, denote

$$(S\Gamma)_g = \{(x, g, x) \in \Gamma_0 \times G \times \Gamma_0 \mid p(x) \in M^g\}.$$

By $i_g : (S\Gamma)_g \rightarrow \Gamma$, we denote the natural inclusion.

Lemma 2.15. *Assume that $\omega \in \Omega^k(\Gamma)$ is a multiplicative k -form, i.e. satisfies $\partial\omega = 0$. Then $i_g^*\omega = 0$.*

PROOF. For simplicity, we prove the lemma for the case $k = 1$. Differential forms of higher degree can be proved in a similar manner. Fix any tangent vector $\delta_x \in T_x p^{-1}(M^g)$. Let $\delta_m = p_*\delta_x \in T_m M$, where $m = p(x)$, and $G^{\delta_m} = \{h \in G \mid R_{h*}\delta_m = \delta_m\}$, the isotropy group at δ_m of the lifted G -action R_{h*} on TM . For any $h \in G^{\delta_m}$, $(\delta_x, 0_h, \delta_x)$ is clearly a well defined tangent vector in $T_{(x, h, x)}\Gamma$, where $0_h \in T_h G$ is the zero tangent vector. Since G^{δ_m} is compact, then $(\delta_x, 0_h, \delta_x) \lrcorner \omega$, considered as a function on G^{δ_m} , must be bounded. On the other hand, it is simple to see that, with respect to the tangent groupoid multiplication $T\Gamma \rightrightarrows T\Gamma_0$, we have

$$(\delta_x, 0_g, \delta_x) \cdot (\delta_x, 0_g, \delta_x) = (\delta_x, 0_{g^2}, \delta_x).$$

Since $\partial\omega = 0$, it follows that $2(\delta_x, 0_g, \delta_x) \lrcorner \omega = (\delta_x, 0_{g^2}, \delta_x) \lrcorner \omega$. Hence, for any $n \in \mathbb{N}^*$,

$$(\delta_x, 0_g, \delta_x) \lrcorner \omega = \frac{1}{n}(\delta_x, 0_{g^n}, \delta_x) \lrcorner \omega.$$

Since $g^n \in G^{\delta_m}$, it thus follows that $(\delta_x, 0_g, \delta_x) \lrcorner \omega = 0$. Hence $i_g^*\omega = 0$. \square

Lemma 2.16. *The groupoid $\Lambda\Gamma|_{(S\Gamma)_g} \rightrightarrows (S\Gamma)_g$ is Morita equivalent to the transformation groupoid $M^g \rtimes G^g \rightrightarrows M^g$.*

PROOF. Consider

$$\begin{array}{ccc} \Lambda\Gamma|_{(S\Gamma)_g} & \xrightarrow{\varphi} & M^g \rtimes G^g \\ \Downarrow & & \Downarrow \\ (S\Gamma)_g & \xrightarrow{\varphi_0} & M^g \end{array}$$

Here the map $\varphi : \Lambda\Gamma|_{(S\Gamma)_g} \rightarrow M^g \rtimes G^g$ is defined by $\varphi((x, g, x), (x, h, z)) = (p(x), h)$, $\forall ((x, g, x), (x, h, z)) \in S\Gamma \rtimes \Gamma|_{(S\Gamma)_g}$, and $\varphi_0 : (S\Gamma)_g \rightarrow M^g$ is $\varphi_0(x, g, x) = p(x)$, $\forall (x, g, x) \in (S\Gamma)_g$. It is simple to check that the diagram above indeed defines a Morita morphism. \square

Proof of Theorem 2.14

- (1). Note that for any $g \in G$, the S^1 -bundle $P^g \rightarrow M^g$ is induced from the S^1 -bundle $\tilde{\Gamma}|_{(S\Gamma)_g} \rightarrow (S\Gamma)_g$ over $\Lambda\Gamma|_{(S\Gamma)_g}$. According to Proposition 2.5, the S^1 -central extension

$\tilde{\Gamma} \rightarrow \Gamma$ admits a connection $\theta \in \Omega^1(\tilde{\Gamma})$. Let $\omega \in \Omega^2(\Gamma)$ be the curvature of the S^1 -bundle $\tilde{\Gamma} \rightarrow \Gamma$, i.e. $d\theta = \pi^*\omega$. Since $\partial\theta = 0$, it follows that $\partial\omega = 0$. Hence Lemma 2.15 implies that $\tilde{\Gamma}|_{(S\Gamma)_g} \rightarrow (S\Gamma)_g$ must be a flat bundle over $\Lambda\Gamma|_{(S\Gamma)_g} \rightrightarrows (S\Gamma)_g$. Therefore $P^g \rightarrow M^g$ is a flat bundle over the transformation groupoid $M^g \rtimes G^g \rightrightarrows M^g$, according to Corollary 3.15 in [17].

Assume that $\theta' \in \Omega^1(\tilde{\Gamma})$ is another connection of the central extension $\tilde{\Gamma} \rightarrow \Gamma \rightrightarrows \Gamma_0$. Then $\theta - \theta' = \pi^*\xi$, where $\xi \in \Omega^1(\Gamma)$ satisfies the equation $\partial\xi = 0$. Applying Lemma 2.15 again, we obtain that $\xi|_{(S\Gamma)_g} = 0$. It thus follows that $\theta' = \theta$ when being restricted to $\tilde{\Gamma}|_{(S\Gamma)_g}$.

By construction, for any $u \in M^g$, $P_u^g = \coprod_{\{x|p(x)=u\}} \tilde{\Gamma}|_{(x,g,x)} / \sim$, where \sim is the equivalence relation between elements in $\tilde{\Gamma}|_{(x,g,x)}$ and those in $\tilde{\Gamma}|_{(y,g,y)}$ with $p(x) = p(y) = u$ induced by the action of the element $(x, 1, y) \in \Gamma$ as given by Eq. (10). To prove that g acts on P^g by the identity map, it suffices to show that $\gamma := (x, g, x)$ acts on P^g by the identity. The latter is equivalent to the identity:

$$(11) \quad \tilde{\gamma}\xi\tilde{\gamma}^{-1} = \xi,$$

for any $\tilde{\gamma} \in \tilde{\Gamma}|_{(x,g,x)}$ which lifts $\gamma = (x, g, x)$, and any $\xi \in \tilde{\Gamma}|_{(x,g,x)}$. Since $\tilde{\gamma}$ and ξ lie in the same fiber $\tilde{\Gamma}|_{(x,g,x)}$, we may assume that $\xi = \lambda\tilde{\gamma}$, $\lambda \in \mathbb{C}^*$. Eq. (11) follows immediately since $\tilde{\Gamma} \rightarrow \Gamma$ is an S^1 -central extension.

(2). To prove (ii), assume that $\tilde{\Gamma}' \xrightarrow{\pi'} \Gamma' \rightrightarrows \Gamma'_0$ is another S^1 -central extension representing α , where $p' : \Gamma'_0 \rightarrow M$ is a surjective submersion and $\Gamma' \rightrightarrows \Gamma'_0$ is the pullback groupoid of the transformation groupoid $M \rtimes G \rightrightarrows M$ by p' . Let $\Gamma''_0 = \Gamma_0 \times_M \Gamma'_0$. By $\text{pr}_1 : \Gamma''_0 \rightarrow \Gamma_0$ and $\text{pr}_2 : \Gamma''_0 \rightarrow \Gamma'_0$, we denote the natural projections. Let $\Gamma'' \rightrightarrows \Gamma''_0$ be the pull back groupoid of the transformation groupoid $M \rtimes G \rightrightarrows M$ via $p'' : \Gamma''_0 \rightarrow M$. By assumption, $\text{pr}_1^*(\tilde{\Gamma} \rightarrow \Gamma \rightrightarrows \Gamma_0)$ and $\text{pr}_2^*(\tilde{\Gamma} \rightarrow \Gamma \rightrightarrows \Gamma_0)$ are Morita equivariant S^1 -central extensions. Since $p'' = p' \circ \text{pr}_2 = p \circ \text{pr}_1$, one finds that $\Gamma'' \rightrightarrows \Gamma''_0$ is isomorphic to the pullback groupoids $\text{pr}_1^*(\Gamma \rightrightarrows \Gamma_0)$ and $\text{pr}_2^*(\Gamma' \rightrightarrows \Gamma'_0)$. Therefore, we obtain two Morita equivariant S^1 -central extensions: $\text{pr}_1^*\tilde{\Gamma} \rightarrow \Gamma'' \rightrightarrows \Gamma''_0$ and $\text{pr}_2^*\tilde{\Gamma}' \rightarrow \Gamma'' \rightrightarrows \Gamma''_0$. By Proposition 4.16 in [5], there exists an S^1 -bundle $L \rightarrow \Gamma''_0$ such that $\text{pr}_2^*\tilde{\Gamma}' \cong L \otimes \text{pr}_1^*\tilde{\Gamma} \otimes L^{-1}$. It thus follows that $\text{pr}_2^*\tilde{\Gamma}'|_{(S\Gamma')_g} \cong \text{pr}_1^*\tilde{\Gamma}|_{(S\Gamma)_g}$. This implies that $\tilde{\Gamma}'|_{(S\Gamma')_g} \rightarrow (S\Gamma')_g$, as an S^1 -bundle over $\Lambda\Gamma'|_{(S\Gamma')_g} \rightrightarrows (S\Gamma')_g$, is isomorphic to $\tilde{\Gamma}|_{(S\Gamma)_g} \rightarrow (S\Gamma)_g$, as an S^1 -bundle over $\Lambda\Gamma|_{(S\Gamma)_g} \rightrightarrows (S\Gamma)_g$.

Remark 2.17. Freed-Hopkins-Teleman also proved the existence of a family of flat S^1 -bundles over M^g using a different method [14].

2.5. Localized twisted equivariant cohomology. Let us first recall some basic constructions of Block-Getzler [8]. Following [8], by a local equivariant differential form on M , we mean a smooth germ at $0 \in \mathfrak{g}$ of a smooth map from \mathfrak{g} to $\Omega^\bullet(M)$ equivariant under the G -action. Denote the space of all local equivariant differential forms by $\bar{\Omega}_G^\bullet(M)$, i.e. $\bar{\Omega}_G^\bullet(M) := C_0^\infty(\mathfrak{g}, \Omega^\bullet(M))^G$. Here for a finite-dimensional vector space V , $C_0^\infty(V)$ denotes the algebra of germs at $0 \in V$ of smooth functions on V . It is clear that $\bar{\Omega}_G^\bullet(M)$ is $\mathbb{Z}/2$ -graded, and is a module over the algebra $C_0^\infty(\mathfrak{g})^G$ of germs of invariant smooth functions over \mathfrak{g} . The usual equivariant differential $d_G = d + \iota : \Omega_G^\bullet(M) \rightarrow \Omega_G^{\bullet+1}(M)$ extends to a differential, denoted by the same symbol d_G ,

$$d_G = d + \iota : \bar{\Omega}_G^\bullet(M) \rightarrow \bar{\Omega}_G^{\bullet+1}(M)$$

satisfying $d_G^2 = 0$. Thus we obtain a $\mathbb{Z}/2$ -graded chain complex $(\bar{\Omega}_G^\bullet(M), d_G)$, which can be considered as a certain completion of the Cartan model of the equivariant cohomology $H_G^\bullet(M)$.

Now let G act on the manifold underlying G by conjugation: $h \cdot g = g^{-1}hg$. For any $g \in G$, by M^g we denote the fixed point set of the diffeomorphism induced by g on M . Let G^g denote the centralizer of g : $G^g = \{h \in G | gh = hg\}$, and \mathfrak{g}^g its Lie algebra. Consider the space of equivariant differential forms $\bar{\Omega}^\bullet(M, G)_g := \bar{\Omega}_{G^g}(M^g)$, which consist of germs at zero of smooth maps from \mathfrak{g}^g to $\Omega(M^g)$ equivariant under G^g . It is easy to see that if $\omega \in \bar{\Omega}^\bullet(M, G)_g$, $k \cdot \omega \in \bar{\Omega}^\bullet(M, G)_{g \cdot k}$. Moreover, the equivariant differential coboundary operators on $\{\bar{\Omega}_{G^g}^\bullet(M^g), g \in G\}$ are compatible with the G action as well. That is

$$k \cdot d_{G^g} \omega = d_{G^{Ad_k g}}(k \cdot \omega),$$

where $d_{G^g} : \bar{\Omega}^\bullet(M, G)_g \rightarrow \bar{\Omega}^\bullet(M, G)_g$ is the equivariant differential on $\bar{\Omega}^\bullet(M, G)_g$. The family of cohomology groups $H^\bullet(\bar{\Omega}^\bullet(M, G)_g, d_{G^g})$ are called *localized equivariant cohomology*.

For any $\alpha \in H_G^3(M, \mathbb{Z})$, let $\coprod_{g \in G} P^g$ be a family of G -equivariant flat S^1 -bundles as in Theorem 2.14, and $L = \coprod_{g \in G} L^g$, where $L^g = P^g \times_{S^1} \mathbb{C}$, $\forall g \in G$, are their associated G -equivariant flat complex line bundles.

Choose a G -equivariant closed 3-form $\eta_G \in Z_G^3(M)$ such that $[\eta_G] = \alpha$. Following Block-Getzler [8], we consider the localized twisted equivariant cohomology as follows. Denote by $\bar{\Omega}^\bullet(M, G, L)_g$ the space $\bar{\Omega}_{G^g}(M^g, L^g)$ of germs at zero of G^g -equivariant smooth maps from \mathfrak{g}^g to $\Omega^\bullet(M^g, L^g)$. By $d_{G^g}^\alpha$ we denote the twisted equivariant differential operator $\nabla_g + \iota - 2\pi i \eta_G$ on $\bar{\Omega}_{G^g}(M^g, L^g)$, where $\nabla_g : \bar{\Omega}_{G^g}(M^g, L^g) \rightarrow \bar{\Omega}_{G^g}(M^g, L^g)$ denotes the covariant differential induced by the flat connection on the complex line bundle $L^g \rightarrow M^g$, and η_G acts on $\bar{\Omega}_{G^g}(M^g, L^g)$ by taking the wedge product with $i_g^* \eta_G$. Here $i_g^* : \Omega_G(M) \rightarrow \Omega_{G^g}(M^g)$ is the restriction map. It is simple to see that $(d_{G^g}^\alpha)^2 = 0$, and $\{d_{G^g}^\alpha | g \in G\}$ are compatible with the G -action. The family of cohomology groups are denoted by $H^\bullet(\bar{\Omega}^\bullet(M, G, L)_g, d_{G^g}^\alpha)$, and are called *localized twisted equivariant cohomology*.

The proposition below justifies our definition.

Proposition 2.18. *Assume that η_G^1 and η_G^2 are equivariant closed 3-forms in $Z_G^3(M)$ such that $\eta_G^2 - \eta_G^1 = d_G B_G$, for some $B_G \in \Omega_G^2(M)$. Then*

$$\begin{aligned} \Phi_{B_G} : \bar{\Omega}^\bullet(M, G, L)_g &\rightarrow \bar{\Omega}^\bullet(M, G, L)_g \\ \omega &\mapsto \exp(2\pi i B_G) \omega \end{aligned}$$

defines an isomorphism of the cochain complexes from $(\bar{\Omega}^\bullet(M, G, L)_g, \nabla_g + \iota - 2\pi i \eta_G^1)$ to $(\bar{\Omega}^\bullet(M, G, L)_g, \nabla_g + \iota - 2\pi i \eta_G^2)$.

Moreover, if B_G is a coboundary, then Φ_{B_G} induces the identity map on the cohomology group $H^\bullet(\bar{\Omega}^\bullet(M, G, L)_g, d_{G^g}^\alpha)$. As a consequence, there is a canonical map

$$(12) \quad H_G^2(M) \rightarrow \text{Aut}(H^\bullet(\bar{\Omega}^\bullet(M, G, L)_g, d_{G^g}^\alpha)).$$

PROOF. The proof of the first part is straightforward. We note that the exponential does make sense, since, if $B_G = \beta + f \in \Omega^2(M)^G \oplus (\mathfrak{g}^* \otimes \Omega^0(M))^G$, $\exp(2\pi i f) \in C^\infty(\mathfrak{g}, M)$ is well-defined and $\exp(2\pi i \beta)$ is a finite sum: $\sum_{k \leq \dim M/2} \frac{(2\pi i)^k}{k!} \beta^k$.

For the second part, assume that $B_G = d_G \gamma$, with $\gamma \in \Omega^1(M)^G$. Then $\eta_G^1 = \eta_G^2$. A simple calculation (see e.g. [28, Prop. 4.8]) shows that, for any cocycle $\omega \in \bar{\Omega}^\bullet(M, G, L)_g$, we have

$$\Phi_{B_G} \omega - \omega = (\nabla + \iota - 2\pi i \eta_G^1)(u\omega),$$

where $u = \gamma \frac{e^{2\pi i B_G} - 1}{B_G}$. \square

As a consequence, the localized twisted equivariant cohomology $H^\bullet(\bar{\Omega}^\bullet(M, G, L)_g, d_{G^g}^\alpha)$ depends on L up to an isomorphism.

3. THE HOCHSCHILD-KOSTANT-ROSENBERG THEOREM

3.1. Equivariant cyclic homology. The Connes' Hochschild-Kostant-Rosenberg theorem states that if M is a compact manifold, then

$$\tau : a_0 \otimes \cdots \otimes a_k \mapsto \frac{1}{k!} a_0 da_1 \cdots da_k$$

induces an isomorphism from $HP_i(C^\infty(M))$ to $H_{dR}^{i+2\mathbb{Z}}(M, \mathbb{C})$ for $i = 0, 1$. In fact, τ is a chain map from the periodic cyclic chain complex $(PC_\bullet(C^\infty(M)), b + \mathcal{B})$ to the de Rham complex $(\Omega^\bullet(M), d_{DR})$ [12, 13]. In general, we have the following

Proposition 3.1. *Let A be an associative unital algebra over \mathbb{C} , (Ω^\bullet, d) a differential graded algebra, $(\bar{\Omega}^\bullet, d)$ a chain complex, and $\rho : A \rightarrow \Omega^0$ an algebra morphism. Assume that $Tr : \Omega^\bullet \rightarrow \bar{\Omega}^\bullet$ is a chain map, called a (super-)trace. Then the map $\tau : PC_\bullet(A) \rightarrow \bar{\Omega}^\bullet$ given by*

$$\tau : a_0 \otimes \cdots \otimes a_k \mapsto \frac{1}{k!} Tr(\rho(a_0) d\rho(a_1) \cdots d\rho(a_k))$$

is a chain map from $(PC_\bullet(A), b + \mathcal{B})$ to $(\bar{\Omega}^\bullet, d)$.

The goal of this section is to prove a counterpart of the above result for a G -equivariant curved differential graded algebra.

Assume that A is a topological algebra over \mathbb{C} endowed with an action of a compact Lie group G by automorphisms. Let $\tilde{A} = A \oplus \mathbb{C}1$ be its unitization considered as a G -algebra, where G acts on the unit 1 trivially. Set $CC_k^G(A) = C^\infty(G, \tilde{A} \otimes A^{\otimes k})^G$, where G acts on itself by conjugation, and \otimes denotes an appropriate topological tensor product chosen according to the situation. There exist two differentials $b : CC_k^G(A) \rightarrow CC_{k-1}^G(A)$ and $\mathcal{B} : CC_k^G(A) \rightarrow CC_{k+1}^G(A)$, defined, respectively, by

$$\begin{aligned} b(\varphi \otimes a_0 \otimes \cdots \otimes a_k)(g) &= \sum_{i=0}^{k-1} (-1)^i \varphi(g) a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_k \\ &\quad + (-1)^k \varphi(g) (g^{-1} a_k) a_0 \otimes a_1 \otimes \cdots \otimes a_{k-1} \\ \mathcal{B}(\varphi \otimes a_0 \otimes \cdots \otimes a_k)(g) &= \sum_{i=0}^k (-1)^{ki} \varphi(g) 1 \otimes (g^{-1} a_{k-i+1}) \otimes \cdots \\ &\quad \cdots \otimes (g^{-1} a_k) \otimes a_0 \otimes \cdots \otimes a_{k-i}, \\ &\quad \forall \varphi \in C^\infty(G), a_i \in A, i = 1, \dots, k. \end{aligned}$$

It is simple to check that $b^2 = \mathcal{B}^2 = b\mathcal{B} + \mathcal{B}b = 0$. Let $PC_i^G(A) = \bigoplus_{m \in \mathbb{N}} CC_{i+2m}^G(A)$. The G -equivariant periodic cyclic homology of A is defined to be the homology group of the chain complex $(PC_\bullet^G(A), b + \mathcal{B})$:

$$HP_i^G(A) = H_i(PC_\bullet^G(A), b + \mathcal{B}).$$

The following result is due to Brylinski [10, 11].

Proposition 3.2. *Let A be a topological associative algebra, and G a compact Lie group acting on A by automorphisms. Then there is a natural isomomorphism*

$$HP_\bullet^G(A) \cong HP_\bullet(A \rtimes G),$$

where $A \rtimes G := C^\infty(G, A)$ is the crossed product algebra.

We refer to [30] for the general theory of equivariant periodic cyclic homology.

3.2. Traces on curved differential graded G -algebras. We use the standard notation for the Lie derivative: $\mathcal{L}_X a = X \cdot a = \lim_{t \rightarrow 0} \frac{e^{tX} \cdot a - a}{t}$ for all $X \in \mathfrak{g}$.

By an \mathbb{N} -graded G -vector space with a connection, we mean a \mathbb{N} -graded vector space $(\Omega^n)_{n \in \mathbb{N}}$ equipped with a degree preserving G -action whose infinitesimal \mathfrak{g} -action is denoted by \mathcal{L}_X , together with a G -equivariant linear map $\nabla : \Omega^\bullet \rightarrow \Omega^{\bullet+1}$ of degree 1, and a G -equivariant linear map $\iota : \mathfrak{g} \rightarrow \text{Der}^{-1}(\Omega^\bullet)$ satisfying the following identities:

- (i) $\iota_X^2 = 0$ for all $X \in \mathfrak{g}$;
- (ii) $\nabla \iota_X + \iota_X \nabla = \mathcal{L}_X$.

Here $\text{Der}^{-1}(\Omega^\bullet)$ denotes the space of degree -1 derivations on Ω^\bullet . The operators ι_X are called contractions, the operators \mathcal{L}_X are called Lie derivatives, and the operator ∇ is called connection.

A *morphism* between two \mathbb{N} -graded G -spaces with connections is a G -equivariant linear map of degree 0 which interchanges the connections and contractions (and hence the Lie derivatives as well).

Similarly, an \mathbb{N} -graded G -algebra with a connection is a \mathbb{N} -graded topological algebra, which is an \mathbb{N} -graded G -vector space with a connection such that \mathcal{L}_X , ι_X , and ∇ are all derivations of the graded algebra.

Given an \mathbb{N} -graded G -space with a connection (Ω^\bullet, ∇) , and $\Theta \in \Omega^2$, set

$$(13) \quad \eta_G = (\nabla + \iota)\Theta,$$

which is a map from \mathfrak{g} to Ω^\bullet , i.e., $\eta_G(X) = \nabla \Theta + \iota_X \Theta$, $\forall X \in \mathfrak{g}$. In the sequel, we write $\Omega = \nabla \Theta \in \Omega^3$ and $\eta_X = \iota_X \Theta \in \Omega^1$.

Definition 3.3. A curved differential graded G -algebra is an \mathbb{N} -graded G -algebra with a connection (Ω^\bullet, ∇) such that $\nabla^2 = [\Theta, \cdot]$ for some $\Theta \in \Omega^2$ satisfying the properties that $\forall X \in \mathfrak{g}$, $\mathcal{L}_X \Theta = 0$, and $\eta_G(X)$ is central in Ω^\bullet .

For instance, for a graded G -manifold M , the de Rham complex $(\Omega^\bullet(M), d_{\text{DR}})$ is clearly a (curved) differential graded G -algebra with $\Theta = 0$.

A module over a \mathbb{N} -graded G -algebra with a connection (Ω^\bullet, ∇) is a \mathbb{N} -graded G -space with a connection $(\bar{\Omega}^\bullet, d)$ such that $\forall \beta \in \Omega^\bullet$ and $\omega \in \bar{\Omega}^\bullet$,

- (i) $d(\beta\omega) = \nabla \beta \cdot \omega + (-1)^{|\beta|} \beta \cdot d\omega$;
- (ii) $\bar{\iota}_X(\beta\omega) = (\bar{\iota}_X \beta)\omega + (-1)^{|\beta|} \beta \iota_X \omega$.

Definition 3.4. A trace map between a curved differential graded G -algebra $(\Omega^n)_{n \in \mathbb{N}}$ and a \mathbb{N} -graded G -space $(\bar{\Omega}^n)_{n \in \mathbb{N}}$ with a connection, is a morphism $\text{Tr} : \Omega^\bullet \rightarrow \bar{\Omega}^\bullet$ of \mathbb{N} -graded G -spaces with connections such that the following identity holds for a fixed central element $g \in G$:

$$\text{Tr}(\omega_1 \omega_2) = (-1)^{|\omega_1| |\omega_2|} \text{Tr}((g^{-1} \omega_2) \omega_1), \quad \forall \omega_1, \omega_2 \in \Omega^\bullet.$$

Lemma 3.5. Let $(\bar{\Omega}^\bullet, d)$ be a module of a curved differential graded algebra (Z^\bullet, ∇) such that $d^2 = 0$. Assume that $\nabla^2 = [\Theta, \cdot]$ for $\Theta \in Z^2$. Let

$$\bar{C}_G^\bullet(\bar{\Omega}) := C_0^\infty(\mathfrak{g}, \bar{\Omega}^\bullet)^G$$

be the space of smooth germs at $0 \in \mathfrak{g}$ of G -equivariant maps from \mathfrak{g} to $\bar{\Omega}^\bullet$. Then $(\bar{C}_G^\bullet(\bar{\Omega}), d + \bar{\iota} + \eta_G)$ is a chain complex, where the coboundary operator $d + \bar{\iota} + \eta_G$ is defined by

$$(d + \bar{\iota} + \eta_G)(\omega)(X) = d(\omega(X)) + \bar{\iota}_X(\omega(X)) + \eta_G(X) \cdot \omega(X), \quad \forall \omega \in \bar{C}_G^\bullet(\bar{\Omega}), \quad X \in \mathfrak{g}.$$

Here η_G is defined by Eq. (13).

3.3. From equivariant periodic cyclic homology to equivariant cohomology. We are now ready to state the main result of this section.

Theorem 3.6. *Let (Ω^\bullet, ∇) be a curved differential graded G -algebra with $\nabla^2 = [\Theta, \cdot]$, and $(\bar{\Omega}^\bullet, d)$ a \mathbb{N} -graded G -space with a connection. Let $Tr: \Omega^\bullet \rightarrow \bar{\Omega}^\bullet$ be a trace map. Assume that $\bar{\Omega}^\bullet$ is a module over the curved differential graded G -algebra Z^\bullet generated by $\eta_G(X)$, i.e. by $\nabla\Theta$ and $\iota_X\Theta$, $\forall X \in \mathfrak{g}$, such that*

$$Tr(\beta\omega) = \beta Tr\omega, \quad \forall \beta \in Z^\bullet, \quad \omega \in \Omega^\bullet.$$

Then, for any algebra homomorphism $\rho: A \rightarrow \Omega^0$, the map $\tau: PC_\bullet^G(A) \rightarrow \bar{C}_G^\bullet(\bar{\Omega})$ defined by

$$\begin{aligned} & \tau(\varphi \otimes a_0 \otimes \cdots \otimes a_k)(X) \\ &= \varphi(ge^X) \int_{\Delta_k} Tr(\rho(a_0)\nabla^{(t_1, X)}\rho(a_1) \cdots \nabla^{(t_k, X)}\rho(a_k))e^{-\Theta} dt_1 \cdots dt_k, \end{aligned}$$

$\forall \varphi \in C^\infty(G)$, $a_i \in A$, $i = 1, \dots, k$, where $\Delta_k = \{(t_1, \dots, t_k) \mid 0 \leq t_1 \leq \cdots \leq t_k \leq 1\}$ and

$$\nabla^{(t, X)}\beta = e^{-t\Theta}\nabla(e^{-tX}\beta)e^{t\Theta}, \quad \forall \beta \in \Omega^\bullet$$

is a chain map from $(PC_\bullet^G(A), b + \mathcal{B})$ to $(C_G^\bullet(\bar{\Omega}), d + \bar{\iota} + \eta_G)$.

We start with the following

Lemma 3.7. *For all $\beta \in \Omega^0$,*

$$-\frac{\partial}{\partial t}(e^{-t\Theta}(e^{-tX}\beta)e^{t\Theta}) = (\nabla + \iota_X)\nabla^{(t, X)}(\beta).$$

PROOF. We have

$$\begin{aligned} -\frac{\partial}{\partial t}(e^{-t\Theta}(e^{-tX}\beta)e^{t\Theta}) &= e^{-t\Theta}[\Theta, e^{-tX}\beta]e^{t\Theta} + e^{-t\Theta}\mathcal{L}_X(e^{-tX}\beta)e^{t\Theta} \\ &= e^{-t\Theta}\nabla^2(e^{-tX}\beta)e^{t\Theta} + e^{-t\Theta}\iota_X\nabla(e^{-tX}\beta)e^{t\Theta}, \end{aligned}$$

where we have used the identities: $\nabla^2 = [\Theta, \cdot]$ and $\mathcal{L}_X = \iota_X\nabla + \nabla\iota_X$. Now, since $\Omega = \nabla\Theta$ is central by assumption, we have

$$\begin{aligned} \nabla\nabla^{(t, X)}(\beta) &= \nabla(e^{-t\Theta}\nabla(e^{-tX}\beta)e^{t\Theta}) \\ &= -te^{-t\Theta}\Omega\nabla(e^{-tX}\beta)e^{t\Theta} + e^{-t\Theta}\nabla^2(e^{-tX}\beta)e^{t\Theta} \\ &\quad - e^{-t\Theta}\nabla(e^{-tX}\beta)t\Omega e^{t\Theta} \\ &= e^{-t\Theta}\nabla^2(e^{-tX}\beta)e^{t\Theta} - te^{-t\Theta}[\alpha, \nabla(e^{-tX}\beta)]e^{t\Theta} \quad (\text{since } \Omega \text{ is central}) \\ &= e^{-t\Theta}\nabla^2(e^{-tX}\beta)e^{t\Theta}. \end{aligned}$$

Similarly, since $\iota_X\Theta := \eta_X$ is central, we have $\iota_X\nabla^{(t, X)}(\beta) = e^{-t\Theta}\iota_X\nabla(e^{-tX}\beta)e^{t\Theta}$. The result thus follows. \square

Proof of Theorem 3.6

Without loss of generality, we may assume that $A = \Omega^0$ and $\rho = \text{id}$. We compute

$$\begin{aligned}
& (d + \bar{\iota})\tau(\varphi \otimes a_0 \otimes \cdots \otimes a_k)(X) \\
&= \int_{\Delta_k} (d + \bar{\iota}_X) \text{Tr}(\varphi(ge^X) a_0 \nabla^{(t_1, X)} a_1 \cdots \nabla^{(t_k, X)} a_k e^{-\Theta}) dt_1 \cdots dt_k \\
&= \int_{\Delta_k} \text{Tr}((\nabla + \iota_X)(\varphi(ge^X) a_0 \nabla^{(t_1, X)} a_1 \cdots \nabla^{(t_k, X)} a_k e^{-\Theta})) dt_1 \cdots dt_k \\
&= \int_{\Delta_k} \varphi(ge^X) \text{Tr}(\nabla a_0 \nabla^{(t_1, X)} a_1 \cdots \nabla^{(t_k, X)} a_k e^{-\Theta}) dt_1 \cdots dt_k \\
&\quad + \sum_{i=1}^k (-1)^{i+1} \int_{\Delta_k} \varphi(ge^X) \text{Tr}(a_0 \nabla^{(t_1, X)} a_1 \cdots (\nabla + \iota_X) \nabla^{(t_i, X)} a_i \cdots \nabla^{(t_k, X)} a_k e^{-\Theta}) dt_1 \cdots dt_k \\
&\quad + (-1)^k \int_{\Delta_k} \varphi(ge^X) \text{Tr}(a_0 \nabla^{(t_1, X)} a_1 \cdots \nabla^{(t_k, X)} a_k (\nabla + \iota_X) e^{-\Theta}) dt_1 \cdots dt_k \\
&= I + II + III.
\end{aligned}$$

Next we examine each term I, II and III separately. Since $(\nabla + \iota_X)e^{-\Theta} = -e^{-\Theta}(\alpha + \eta_X)$, the third term is equal to

$$\begin{aligned}
III &= -(-1)^k \int_{\Delta_k} \text{Tr}(\varphi(ge^X) a_0 \nabla^{(t_1, X)} a_1 \cdots \nabla^{(t_k, X)} a_k e^{-\Theta}(\alpha + \eta_X)) dt_1 \cdots dt_k \\
&= -(\alpha + \eta_X)\tau(\varphi \otimes a_0 \otimes \cdots \otimes a_k)(X).
\end{aligned}$$

Using Lemma 3.7 above, we see that the second term

$$\begin{aligned}
II &= \int_{(t_1, \dots, \hat{t}_i, \dots, t_k) \in \Delta_{k-1}} \left(\sum_{i=1}^k (-1)^{i+1} \text{Tr}(\varphi(ge^X) a_0 \nabla^{(t_1, X)} a_1 \cdots \nabla^{(t_{i-1}, X)} a_{i-1} (e^{-t_{i-1}\Theta} (e^{-t_{i-1}X} a_i) e^{t_{i-1}\Theta}) \right. \\
&\quad \left. \nabla^{(t_{i+1}, X)} a_{i+1} \cdots e^{-\Theta}) + (-1)^i \text{Tr}(\varphi(ge^X) a_0 \nabla^{(t_1, X)} a_1 \cdots \right. \\
&\quad \left. \nabla^{(t_{i-1}, X)} a_{i-1} (e^{-t_{i+1}\Theta} (e^{-t_{i+1}X} a_i) e^{t_{i+1}\Theta}) \nabla^{(t_{i+1}, X)} a_{i+1} \cdots e^{-\Theta}) \right) dt_1 \cdots \widehat{dt_i} \cdots dt_k
\end{aligned}$$

with $t_0 = 0$ and $t_{k+1} = 1$ by convention.

After re-indexing, we obtain

$$\begin{aligned}
II &= \int_{(t_1, \dots, t_{k-1}) \in \Delta_{k-1}} \sum_{i=1}^k ((-1)^{i+1} \text{Tr}(\varphi(ge^X) a_0 \nabla^{(t_1, X)} a_1 \cdots \nabla^{(t_{i-1}, X)} a_{i-1} e^{-t_{i-1}\Theta} \\
&\quad \nabla(e^{-t_{i-1}X} a_{i-1})(e^{-t_{i-1}X} a_i) e^{t_{i-1}\Theta} \cdots \nabla^{(t_{k-1}, X)} a_k e^{-\Theta}) dt_1 \cdots dt_{k-1} \\
&\quad + (-1)^i \text{Tr}(\varphi(ge^X) a_0 \nabla^{(t_1, X)} a_1 \cdots \nabla^{(t_{i-1}, X)} a_{i-1} e^{-t_i\Theta} (e^{-t_iX} a_i) \nabla(e^{-t_iX} a_{i+1}) e^{t_i\Theta} \cdots \\
&\quad \nabla^{(t_{k-1}, X)} a_k \cdots e^{-\Theta})) dt_1 \cdots dt_{k-1}
\end{aligned}$$

After replacing i by $i + 1$ in the first sum, and using the derivation property for ∇ , we obtain

$$\begin{aligned}
II &= \int_{\Delta_{k-1}} (\text{Tr}(\varphi(ge^X) a_0 a_1 \nabla^{(t_1, X)} a_2 \dots \nabla^{(t_{k-1}, X)} a_k e^{-\Theta}) \\
&\quad + \sum_{i=1}^{k-1} (-1)^i \text{Tr}(\varphi(ge^X) a_0 \nabla^{(t_1, X)} a_1 \dots \nabla^{(t_i, X)} (a_i a_{i+1}) \dots \nabla^{(t_{k-1}, X)} a_k e^{-\Theta}) \\
&\quad + (-1)^k \text{Tr}(\varphi(ge^X) a_0 \nabla^{(t_1, X)} a_1 \dots \nabla^{(t_{k-1}, X)} a_{k-1} e^{-\Theta} (e^{-X} a_k))) dt_1 \dots dt_{k-1} \\
&= \sum_{i=0}^{k-1} (-1)^i \tau(\varphi \otimes a_0 \otimes \dots \otimes a_i a_{i+1} \otimes \dots \otimes a_k) \\
&\quad + (-1)^k \int_{\Delta_{k-1}} \text{Tr}(\varphi(ge^X) (g^{-1} e^{-X} a_k) a_0 \nabla^{(t_1, X)} a_1 \dots \nabla^{(t_{k-1}, X)} a_{k-1}) e^{-\Theta} dt_1 \dots dt_{k-1} \\
&= (\tau \circ b)(\varphi \otimes a_0 \otimes \dots \otimes a_k)(X).
\end{aligned}$$

It remains to show that the first term I is equal to $(\tau \circ \mathcal{B})(\varphi \otimes a_0 \otimes \dots \otimes a_k)(X)$.

$$\begin{aligned}
&(\tau \circ \mathcal{B})(\varphi \otimes a_0 \otimes \dots \otimes a_k)(X) \\
&= \sum_{i=0}^k (-1)^{ki} \int_{\Delta_{k+1}} \text{Tr}(\varphi(ge^X) \nabla^{(t_0, X)} (g^{-1} e^{-X} a_{k-i+1}) \dots \\
&\quad \dots \nabla^{(t_{i-1}, X)} (g^{-1} e^{-X} a_k) \nabla^{(t_i, X)} a_0 \dots \nabla^{(t_k, X)} a_{k-i} e^{-\Theta}) dt_0 \dots dt_k \\
&= \sum_{i=0}^k \int_{\Delta_{k+1}} \text{Tr}(\varphi(ge^X) \nabla^{(t_i, X)} a_0 \dots \\
&\quad \dots \nabla^{(t_k, X)} a_{k-i} \nabla^{(1+t_0, X)} a_{k-i+1} \dots \nabla^{(1+t_{i-1}, X)} a_k e^{-\Theta}) dt_0 \dots dt_k \\
&= \sum_{i=0}^k \int_{\Delta_{k+1}} \text{Tr}(\varphi(ge^X) e^{-t_i X} \cdot (\nabla a_0 \dots \nabla^{(t_k - t_i, X)} a_{k-i} \nabla^{(1+t_0 - t_i, X)} a_{k-i+1} \dots \\
&\quad \dots \nabla^{(1+t_{i-1} - t_i, X)} a_k) e^{-\Theta}) dt_0 \dots dt_k,
\end{aligned}$$

where we used the G -invariance of ∇ .

Change variables: $s'_0 = t_i$, $s_1 = t_{i+1} - t_i$, \dots , $s_{k-i} = t_k - t_i$, $s_{k-i+1} = 1 + t_0 - t_i$, \dots , $s_k = 1 + t_{i-1} - t_i$. One immediately checks that $(t_0, \dots, t_k) \in \Delta_{k+1}$ if and only if $(s_1, \dots, s_k) \in \Delta_k$ and $s_{k-i} \leq 1 - s'_0 \leq s_{k-i+1}$. Thus,

$$\begin{aligned}
&(\tau \circ \mathcal{B})(\varphi \otimes a_0 \otimes \dots \otimes a_k)(X) \\
&= \sum_{i=0}^k \int_{\Delta_k} \int_{s_{k-i} \leq 1 - s'_0 \leq s_{k-i+1}} \text{Tr}(e^{-s'_0 X} \cdot (\varphi(ge^X) \nabla a_0 \nabla^{(s_1, X)} a_1 \dots \nabla^{(s_k, X)} a_k) e^{-\Theta}) ds'_0 ds_1 \dots ds_k.
\end{aligned}$$

Since $\varphi \otimes a_0 \otimes \dots \otimes a_k$ is a G -invariant element by assumption, it follows that $\varphi(ge^X) \nabla a_0 \nabla^{(s_1, X)} a_1 \dots \nabla^{(s_k, X)} a_k$ must be also G -invariant. Therefore the equation above equals

$$\sum_{i=0}^k \int_{\Delta_k} (s_{k-i+1} - s_{k-i}) \text{Tr}(\varphi(ge^X) \nabla a_0 \nabla^{(s_1, X)} a_1 \dots \nabla^{(s_k, X)} a_k e^{-\Theta}) ds_1 \dots ds_k,$$

(with $s_0 = 0$ and $s_{k+1} = 1$ by convention), which in turn is equal to

$$\int_{\Delta_k} \text{Tr}(\varphi(ge^X) \nabla a_0 \nabla^{(s_1, X)} a_1 \cdots \nabla^{(s_k, X)} a_k e^{-\Theta}) ds_1 \cdots ds_k.$$

This concludes the proof of Theorem 3.6.

4. MAIN THEOREM

4.1. Pseudo-etale structure. Let M be a manifold with an action of a compact Lie group G . In this section, we investigate a special kind of G -equivariant bundle gerbes, whose properties are needed for our future study.

Recall that a pseudo-etale structure [4, 23] on a Lie groupoid $H_1 \rightrightarrows H_0$ is an integrable subbundle F of TH_1 such that

- (1) $F \rightrightarrows TH_0$ is a subgroupoid of the tangent Lie groupoid $TH_1 \rightrightarrows TH_0$;
- (2) for all $\gamma \in H_1$, $s_* : F_\gamma \rightarrow T_{s(\gamma)}H_0$ and $t_* : F_\gamma \rightarrow T_{t(\gamma)}H_0$ are isomorphisms;
- (3) $F|_{H_0} = TH_0$.

Definition 4.1. (1) Let $f : N \rightarrow M$ be a surjective submersion. For any $x \in M$, denote by N_x the fiber $f^{-1}(x)$. A fiberwise measure is a family $(\mu_x)_{x \in M}$ of measures such that for all x , the support of μ_x is a subset of N_x .
 (2) A fiberwise measure $\mu = (\mu_x)_{x \in M}$ is said to be smooth if for all $f \in C_c^\infty(N)$, the map

$$x \mapsto \int f d\mu_x$$

is smooth.

Proposition 4.2. *Let $p : M' \rightarrow M$ be an immersion such that*

- (1) $\sigma : M' \times G \rightarrow M$, $\sigma(x, g) = p(x)g$ is a surjective submersion, and
- (2) $\forall x, y \in M'$ and $g \in G$ satisfying $p(x) = p(y)g$, there exists a diffeomorphism ϕ from a neighborhood U_x of x in M' to a neighborhood U_y of y in M' , which is compatible with the diffeomorphism on M induced by the action by g . That is,

$$p(\phi(z)) = p(z)g, \quad \forall z \in U_x.$$

Let $H_0 = M' \times G$, and $H_1 = H_0 \times_M H_0$. Then

- (1) there is a G -invariant pseudo-etale structure on the Lie groupoid $H_1 \rightrightarrows H_0$;
- (2) there exists a G -invariant Haar system $(\lambda^x)_{x \in H_0}$ on the Lie groupoid $H_1 \rightrightarrows H_0$.

PROOF. (1) For any $((x, g), (y, h)) \in H_1$, where $x, y \in M'$ and $g, h \in G$, set

$$F_{((x, g), (y, h))} = \{((u, l_g X), (v, l_h X)) | \forall u \in T_x M', v \in T_y M', X \in \mathfrak{g} \text{ such that } R_{g*} p_* u = R_{h*} p_* v\}. \blacksquare$$

It is simple to see that F is a G -invariant subbundle of TH_1 , and is a Lie subgroupoid of $TH_1 \rightrightarrows TH_0$. To prove that $s_* : F_\gamma \rightarrow T_{s(\gamma)}H_0$ is an isomorphism, it suffices to show that, for any $((x, g), (y, h)) \in H_0 \times_M H_0$, and any $u \in T_x M'$ and $X \in \mathfrak{g}$, there is a unique $v \in T_y M'$ such that $R_{g*} p_* u = R_{h*} p_* v$. This holds due to the fact that

$$R_{\gamma*}(p_*(T_x M')) = p_*(T_y M'), \text{ when } p(x)\gamma = p(y), \quad x, y \in M', \gamma \in G,$$

which follows from Assumption (2). To see that F is integrable, note that for any $((x, g), (y, h)) \in H_1$, the submanifold $\{(z, g\gamma), (\phi(z), h\gamma) | z \in U_x, \gamma \in G\}$ is a leaf of F in H_1 through this point.

(2) It is simple to see that Haar systems on the Lie groupoid $H_1 \rightrightarrows H_0$ are in one-one correspondence with fiberwise smooth measures of the map $\sigma : H_0 \rightarrow M$, and G -invariant Haar systems correspond to a G -invariant fiberwise smooth measures. Such a measure always exists since G is compact. \square

Example 4.3. Let (U_i) be an open cover of M , and $p : M' = \coprod U_i \rightarrow M$ the covering map. It is clear that the assumption in Proposition 4.2 is satisfied.

4.2. Statement of the main theorem. Let $f : N \rightarrow M$ be a surjective submersion. For any $x \in M$, denote by N_x the fiber $f^{-1}(x)$. Assume that F is a horizontal distribution for $f : N \rightarrow M$, i.e. a subbundle $F \subseteq TN$ satisfying the condition that for all $y \in N$, $f_* : F_y \rightarrow T_{f(y)}M$ is an isomorphism. For any vector field X on M , denote by \tilde{X} its horizontal lifting on N , and by Φ_t the flow of \tilde{X} . Note that the flow preserves fibers. More precisely, for any $x \in M$ and any compact subset $K \subseteq N_x$, if $|t|$ is small enough, then Φ_t is well defined on K and maps K to N_y for some $y \in M$.

We say that a fiberwise smooth measure $(\mu_x)_{x \in M}$ of $f : N \rightarrow M$ is preserved by F if for any vector field X on M , any $x \in M$, $f \in C_c^\infty(N_x)$, $t > 0$, $y \in M$ such that Φ_t is well defined on the support of f and maps it to N_y , the equality

$$\int f d\mu_x = \int f \circ (\Phi_t^X)^{-1} d\mu_y$$

holds.

Note that a pseudo-etale structure on a Lie groupoid $H_1 \rightrightarrows H_0$ induces an action of the Lie groupoid $H_1 \rightrightarrows H_0$ on the vector bundle $TH_0 \rightarrow H_0$.

Definition 4.4. Let M be a G -space. An immersion $p : M' \rightarrow M$ is said to be *nice* if it satisfies the assumptions in Proposition 4.2, and, in addition, there exists a G -invariant integrable horizontal distribution F' for the surjective submersion $\sigma : H_0 := M' \times G \rightarrow M$ and a G -invariant fiberwise smooth measure $(\mu_x)_{x \in M}$ for $\sigma : H_0 \rightarrow M$ such that

- (1) F' is preserved under the action of $H_1 \rightrightarrows H_0$ induced by the pseudo-etale structure $F \subset TH_1$;
- (2) $(\mu_x)_{x \in M}$ is preserved by F' .

Lemma 4.5. Assume that M is a G -space, and $p : M' \rightarrow M$ is a *nice* immersion. Equip a G -invariant Haar system on $H_1 \rightrightarrows H_0$ as in Proposition 4.2. For any vector field X on M , denote by Y its horizontal lift to H_0 tangent to F' , and by \tilde{X} the corresponding vector field on H_1 tangent to F such that $t_*(\tilde{X}_h) = Y_{t(h)}$, $\forall h \in H_1$. Then the flow of \tilde{X} maps a t -fiber to another t -fiber, and preserves the Haar system.

PROOF. Denote by φ_t and φ'_t the flows of \tilde{X} and Y respectively. Since $t_*(\tilde{X}_h) = Y_{t(h)}$ for all $h \in H_1$, we have $\varphi_t(h) \in H_1^{\varphi'_t(x)}$ for all $h \in H_1^x$.

Moreover, since $s_*(\tilde{X}_h) = Y_{s(h)}$, the following diagram commutes:

$$\begin{array}{ccc} H_1^x & \xrightarrow{s} & H_0|_{\sigma(x)} \\ \downarrow \varphi_t & & \downarrow \varphi'_t \\ H_1^{\varphi'_t(x)} & \xrightarrow{s} & H_0|_{\sigma(\varphi'_t(x))}. \end{array}$$

The conclusion follows easily. \square

Lemma 4.6. Let M be a G -space, (U_i) any open cover of M , and $p : M' := \coprod U_i \rightarrow M$ the covering map. Then $p : M' \rightarrow M$ is a *nice* immersion.

PROOF. In this case, $H_0 = \coprod U_i \times G$. It is easy to see that for any $(x, g) \in U_i \times G$,

$$F'_{x,g} = \{(v, 0) | v \in T_x U_i\} \subset T_{(x,g)} H_0$$

is a G -invariant integrable horizontal distribution for the surjective submersion $\sigma : H_0 \rightarrow M$. Define a G -invariant fiberwise smooth measure $(\mu_x)_{x \in M}$ for $\sigma : H_0 \rightarrow M$ as follows.

For any $x \in M$,

$$\sigma^{-1}(x) = \coprod_i \{(xg^{-1}, g) | \forall g \in G, xg^{-1} \in U_i\}$$

For $f \in C_c^\infty(\sigma^{-1}(x))$,

$$\int f d\mu_x = \sum_i \int_G f(xg^{-1}, g) d\lambda^G,$$

where λ^G is the right invariant Haar measure on G . It is simple to check that, equipped with the above structures, σ is indeed a nice immersion. \square

Now assume that $\alpha \in H_G^3(M, \mathbb{Z}) \cong H^3((M \rtimes G)_\bullet, \mathbb{Z})$ is an equivariant integer class. Let $p : \coprod U_i \rightarrow M$ be an open cover of M such that α is represented by an S^1 -central extension $\tilde{\Gamma} \xrightarrow{\pi} \Gamma \rightrightarrows \coprod U_i$. Then, according to Lemma 4.6, $p : M' := \coprod U_i \rightarrow M$ is a nice immersion. Let $H_0 = M' \times G$, $H_1 = H_0 \times_M H_0$, and $\tilde{H}_1 = H_1 \times_\Gamma \tilde{\Gamma}$. Then, according to Corollary 2.10, $\tilde{H}_1 \rightarrow H_1 \rightrightarrows H_0$ is a G -equivariant bundle gerbe over M such that the Dixmier-Douady class of $\tilde{H}_1 \rtimes G \xrightarrow{\phi} H_1 \rtimes G \rightrightarrows H_0$ is equal to α under the natural isomorphism

$$H^3((H \rtimes G)_\bullet, \mathbb{Z}) \cong H^3((M \rtimes G)_\bullet, \mathbb{Z}) \cong H_G^3(M, \mathbb{Z}).$$

Indeed, according to Theorem 2.8, we can choose an equivariant connection $\theta \in \Omega^1(\tilde{H}_1)^G$ and an equivariant curving $B_G \in \Omega_G^2(H_0)$, whose equivariant 3-curvature $\eta_G \in \Omega_G^3(M)$ represents α in the Cartan model. Write

$$(14) \quad B_G = B + \lambda, \quad B \in \Omega^2(H_0) \text{ and } \lambda \in (\mathfrak{g}^* \otimes C^\infty(H_0))^G.$$

Moreover, we can assume that $\theta \in \Omega^1(\tilde{H}_1)^G$ is G -basic. Hence $pr_1^* \theta \in \Omega^1(\tilde{H}_1 \rtimes G)$ is a connection for the S^1 -central extension $\tilde{H}_1 \rtimes G \xrightarrow{\phi} H_1 \rtimes G \rightrightarrows H_0$ according to Lemma 3.3 and Remark 3.5 [27]. By Theorem 2.14, there is a family of G -equivariant flat S^1 -bundles $P^g \rightarrow M^g$ indexed by $g \in G$ such that the action by h is an isomorphism of flat bundles $(P^g \rightarrow M^g) \mapsto (P^{h^{-1}gh} \rightarrow M^{h^{-1}gh})$ over the map $R_h : M^g \rightarrow M^{h^{-1}gh}$ and g acts on P^g as an identity map.

Let $L^g = P^g \times_{S^1} \mathbb{C}$ be the associated complex line bundle. Denote, by ∇_g , the induced covariant derivative on $L^g \rightarrow M^g$.

By choosing a G -invariant Haar system $(\lambda^x)_{x \in H_0}$ on the groupoid $H_1 \rightrightarrows H_0$, which always exists according to Proposition 4.2, we construct a convolution algebra $C_c^\infty(H, L)$: it is the space of compact supported sections of the complex line bundle $L \rightarrow H_1$, where $L = \tilde{H}_1 \times_{S^1} \mathbb{C}$, equipped with the convolution product:

$$(\xi * \eta)(\gamma) = \int_{h \in H_1^{t(\gamma)}} \xi(h) \cdot \eta(h^{-1}\gamma) \lambda^{t(\gamma)}(dh), \quad \forall \xi, \eta \in \Gamma_c(L).$$

The following lemma can be easily verified.

Lemma 4.7. $C_c^\infty(H, L)$ is a G -algebra.

The main result of this section is the following

Theorem 4.8. *Under the hypothesis above, there exists a family of G -equivariant chain maps, indexed by $g \in G$:*

$$\tau_g : (PC_\bullet^G(C_c^\infty(H, L)), b + \mathcal{B}) \rightarrow (\bar{\Omega}^\bullet(M, G, L)_g, d_{G^g}^\alpha)$$

By G -equivariant chain maps, we mean that the following diagram of chain maps commutes:

$$\begin{array}{ccc} (PC_{\bullet}^G(C_c^\infty(H, L)), b + \mathcal{B}) & \xrightarrow{\tau_g} & ((\bar{\Omega}^\bullet(M, G, L)_g, d_{G^g}^\alpha)) \\ & \searrow \tau_{h^{-1}gh} & \downarrow \phi_h \\ & & ((\bar{\Omega}^\bullet(M, G, L)_{h^{-1}gh}, d_{G^{h^{-1}gh}}^\alpha)). \end{array}$$

As an immediate consequence, we have the following

Theorem 4.9. *Under the same hypothesis as in Theorem 4.8, there is a family of morphisms on the level of cohomology:*

$$(15) \quad \tau_g : HP_{\bullet}^G(C_c^\infty(H, L)) \rightarrow H^\bullet((\bar{\Omega}^\bullet(M, G, L)_g, d_{G^g}^\alpha))$$

The proof of Theorem 4.8 occupies the next two subsections. The idea is to apply Theorem 3.6. First of all, since, G^g , $\forall g \in G$, is a Lie subgroup of G , there is a natural chain map

$$(16) \quad (PC_{\bullet}^G(C_c^\infty(H, L)), b + \mathcal{B}) \rightarrow (PC_{\bullet}^{G^g}(C_c^\infty(H, L)), b + \mathcal{B}).$$

To define the map τ_g , we need to construct a chain map:

$$(17) \quad (PC_{\bullet}^{G^g}(C_c^\infty(H, L)), b + \mathcal{B}) \rightarrow (\bar{\Omega}^\bullet(M, G, L)_g, d_{G^g}^\alpha)$$

For this purpose, we need, for each fixed $g \in G$,

- (1) to construct a curved differential graded G^g -algebra (Ω^\bullet, ∇) together with an algebra homomorphism $\rho : C_c^\infty(H, L) \rightarrow \Omega^0$;
- (2) a trace map $\text{Tr}_g : \Omega^\bullet \rightarrow \Omega^\bullet(M^g, L^g)$.

Then we need to prove that they satisfy all the compatibility conditions so that we can apply Theorem 3.6 to obtain the desired chain map.

4.3. A curved differential graded G -algebra. In this subsection, we deal with the first issue as pointed out at the end of last subsection. After replacing G by G^g , we may assume that g is central without loss of generality.

Denote by $\tilde{\Omega}^\bullet(H, L)$ the space of smooth sections of the bundle $\wedge^\bullet F^* \otimes L \rightarrow H_1$, and by $\tilde{\Omega}_c^\bullet(H, L)$ the subspace of compactly supported smooth sections. Consider the convolution product on $\tilde{\Omega}_c^\bullet(H, L)$:

$$(\xi * \eta)(\gamma) = \int_{h \in H_1^{t(\gamma)}} \xi(h) \cdot \eta(h^{-1}\gamma) \lambda^{t(\gamma)}(dh).$$

Here, the product of $\xi(h) \in \wedge^k F_h^* \otimes L_h$ with $\eta(h^{-1}\gamma) \in \wedge^l F_{h^{-1}\gamma}^* \otimes L_{h^{-1}\gamma}$ is obtained as follows. The product $L_h \otimes L_{h^{-1}\gamma} \rightarrow L_\gamma$ is induced from the groupoid structure on $\tilde{H}_1 \rightrightarrows H_0$, and $\wedge^k F_h^* \otimes \wedge^l F_{h^{-1}\gamma}^* \rightarrow \wedge^{k+l} F_\gamma^*$ is the composition of the identifications $F_h^* \rightarrow F_\gamma^*$ and $F_{h^{-1}\gamma}^* \rightarrow F_\gamma^*$, via the pseudo-etale structure, with the wedge product. Then $\tilde{\Omega}_c^\bullet(H, L)$ is a G -equivariant graded associative algebra. The reason that we need compactly supported “forms” on H is because the convolution product does not generally make sense on $\tilde{\Omega}^\bullet(H, L)$.

Let

$$\Omega^\bullet = \Omega^\bullet(H_0) \oplus \tilde{\Omega}_c^\bullet(H, L).$$

Introduce a multiplication on Ω^\bullet by the following formula:

$$(\beta_1 \oplus \omega_1) \cdot (\beta_2 \oplus \omega_2) = (\beta_1 \wedge \beta_2) \oplus (t^* \beta_1 \wedge \omega_2 + \omega_1 \wedge s^* \beta_2 + \omega_1 * \omega_2), \quad \forall \beta_1, \beta_2 \in \Omega^\bullet(H_0), \omega_1, \omega_2 \in \tilde{\Omega}_c^\bullet(H, L). \blacksquare$$

Note that $t^* \beta_1$ and $s^* \beta_2$ can be considered as elements in $\tilde{\Omega}^\bullet(H_1) := C^\infty(H_1, \wedge^\bullet F^*)$, of which $\tilde{\Omega}_c^\bullet(H, L)$ is naturally a bimodule.

Lemma 4.10. *Under the product above, Ω^\bullet is a G -equivariant graded associative algebra.*

PROOF. It is straightforward to check the following relations, $\forall \omega_i \in \tilde{\Omega}_c^\bullet(H, L)$, $\beta \in \tilde{\Omega}^\bullet(H_1)$:

$$(18) \quad t^* \beta \wedge (\omega_1 * \omega_2) = (t^* \beta \wedge \omega_1) * \omega_2$$

$$(19) \quad (\omega_1 * \omega_2) \wedge s^* \beta = \omega_1 * (\omega_2 \wedge s^* \beta)$$

$$(20) \quad (\omega_1 \wedge s^* \beta) * \omega_2 = \omega_1 * (t^* \beta \wedge \omega_2).$$

The associativity thus follows immediately. \square

Lemma 4.11. *The covariant derivative $\nabla : \Omega_c^\bullet(H, L) \rightarrow \Omega_c^{\bullet+1}(H, L)$ induces an algebra derivation: $\nabla : \tilde{\Omega}_c^\bullet(H, L) \rightarrow \tilde{\Omega}_c^{\bullet+1}(H, L)$. I.e.,*

$$(21) \quad \nabla(\omega_1 * \omega_2) = \nabla \omega_1 * \omega_2 + (-1)^{|\omega_1|} \omega_1 * \nabla \omega_2 \quad \forall \omega_1, \omega_2 \in \tilde{\Omega}_c^\bullet(H, L).$$

PROOF. Since the pseudo-etale structure $F \subseteq TH$ is an integrable distribution, by restriction, the connection on $L \rightarrow H$ induces an F -connection, as a Lie algebroid connection, on L . Therefore we have the induced covariant derivative $\nabla : \tilde{\Omega}_c^\bullet(H, L) \rightarrow \tilde{\Omega}_c^{\bullet+1}(H, L)$.

We now prove Eq. (21) by induction on $n := |\omega_1| + |\omega_2|$. If $n = 0$, this is automatically true since ∇ is a connection. Assume (21) is valid for $n = k$. Since $t_* : F_x \rightarrow T_{t(x)} H_0$ is an isomorphism, we can assume that locally $\omega_1 = t^*(df) \wedge \eta$, where $f \in C^\infty(H_0)$ and $\eta \in \Omega_c^{|\omega_1|-1}(H, L)$. Using Eq. (18) and the induction hypothesis, we have

$$\begin{aligned} \nabla(\omega_1 * \omega_2) &= \nabla(t^*(df) \wedge (\eta * \omega_2)) \\ &= -t^*(df) \wedge \nabla(\eta * \omega_2) \\ &= -t^*(df) \wedge (\nabla \eta * \omega_2 + (-1)^{|\eta|} \eta * \nabla \omega_2) \\ &= \nabla(t^*(df) \wedge \eta) * \omega_2 + (-1)^{|\omega_1|} ((t^* df) \wedge \eta) * \nabla \omega_2. \end{aligned}$$

Hence Eq. (21) follows. \square

Extend ∇ to Ω^\bullet by

$$(22) \quad \nabla(\beta \oplus \omega) = d\beta \oplus \nabla \omega, \quad \forall \omega \in \tilde{\Omega}_c^\bullet(H, L), \quad \beta \in \Omega^\bullet(H_0).$$

The following lemma is straightforward:

Lemma 4.12. $\nabla : \Omega^\bullet \rightarrow \Omega^{\bullet+1}$ is a degree +1 derivation.

Lemma 4.13. The map $\nabla : \Omega^\bullet \rightarrow \Omega^{\bullet+1}$ satisfies

$$\nabla^2 = -2\pi i[B, \cdot],$$

where $B \in \Omega^2(H_0) \subset \Omega^\bullet$ is defined as in Eq. (14).

PROOF. The relation $\nabla^2 \beta = -2\pi i[B, \beta]$ holds automatically for $\beta \in \Omega^\bullet(H_0)$ since both sides are zero. For $\omega \in \tilde{\Omega}_c^\bullet(H, L)$, we have

$$\nabla^2 \omega = 2\pi i d\theta \wedge \omega = 2\pi i \partial B \wedge \omega = -2\pi i (t^* B \wedge \omega - s^* B \wedge \omega) = -2\pi i [B, \omega].$$

The conclusion thus follows. \square

Proposition 4.14. (Ω^\bullet, ∇) is a curved differential graded G -algebra.

PROOF. Define the map ι_X , $\forall X \in \mathfrak{g}$, on $\Omega^\bullet(H_0)$ by the usual contraction. Since the pseudo-etale structure $F \subset TH_1$ is preserved by the group G -action, it follows that the contraction map ι_X is also defined on $\Omega_c^\bullet(H, L)$. Thus we have a map $\iota_X : \Omega^\bullet \rightarrow \Omega^{\bullet-1}$ satisfying $\iota_X^2 = 0$. One proves, by induction similar to the proof of Lemma 4.11, that ι_X is indeed a derivation.

It remains to prove that $\nabla \iota_X + \iota_X \nabla = \mathcal{L}_X$ holds. This is clearly true on the component $\Omega^\bullet(H_0)$. Since both sides are derivations on $\tilde{\Omega}_c^\bullet(H, L)$ and this holds for elements in $\tilde{\Omega}^\bullet(H) := C^\infty(H_1, \wedge^\bullet F^*)$, it suffices to prove this identity for elements of degree zero, i.e. $\mathcal{L}_X \xi = \nabla_X \xi$ for all $X \in \mathfrak{g}$ and $\xi \in C_c^\infty(H_1, L)$. Note that for a G -equivariant complex line bundle we always have the identity $\nabla_X - \mathcal{L}_X = \iota_X \theta$. Here $\iota_X \theta$ can be considered as a function on H_0 .

Since θ is G -basic, it follows that $\mathcal{L}_X \xi = \nabla_X \xi$. This concludes the proof. \square

4.4. Trace map. Now we deal with the second issue at the end of Section 4.2.

Consider $\bar{\Omega}^n = \Omega_c^n(M^g, L^g)$. Let $d : \bar{\Omega}^n \rightarrow \bar{\Omega}^{n+1}$ denote the covariant derivative $\Omega_c^n(M^g, L^g) \rightarrow \Omega_c^{n+1}(M^g, L^g)$ of the flat complex line bundle $L^g \rightarrow M^g$. It is clear that $\bar{\Omega}^\bullet$ admits a G^g -action. Let $\bar{\iota} : \bar{\Omega}^n \rightarrow \bar{\Omega}^{n-1}$ be the usual contraction.

Lemma 4.15. $(\bar{\Omega}, d)$ is a \mathbb{N} -graded G -vector space with a connection.

For any fixed $g \in G$, since G^g is a Lie subgroup of G , (Ω^\bullet, ∇) is a curved G^g -differential algebra. Next we will introduce a trace map

$$\text{Tr}_g : \Omega^\bullet \rightarrow \bar{\Omega}^\bullet$$

On the first component $\Omega^\bullet(H_0)$, we set Tr_g to be zero. Before we define Tr_g on the second component of Ω^\bullet , some remarks are in order. Denote $(S(H_1 \rtimes G))_g$ by H_1^g . I.e.

$$H_1^g = \{(x, xg^{-1}) \mid \sigma(x) \in M^g\}.$$

Note that the subbundle $F' \subset TH_0$ corresponds to a subbundle $F_1 \subset F$ such that $t_*(F_1)_\gamma = F'_{t(\gamma)}$ for all $\gamma \in H_1$. Let $F_1^g = F_1 \cap TH_1^g$. Then $(\sigma \circ s)_* = (\sigma \circ t)_* : (F_1)_\gamma \rightarrow TM_{(\sigma \circ s)(\gamma)}^g$ is an isomorphism for all $\gamma \in H_1^g$. Denote by $i^g : \wedge F^* \rightarrow \wedge (F_1^g)^*$ the restriction map.

We define, for any fixed γ ,

$$(\text{Tr}_g \omega)_\gamma = i^g \int_{h \in H_1^g} h_* \omega_{h^{-1}\gamma h g^{-1}} \lambda^x(dh) \in \wedge (F_1^g)_\gamma^* \otimes L|_\gamma.$$

If we write $\gamma = (x, xg^{-1})$, $x \in H_0^g$, and $h = (x, y)$, then

$$h^{-1}\gamma h g^{-1} = (y, x)(x, xg^{-1})(xg^{-1}, yg^{-1}) = (y, yg^{-1}).$$

Therefore $\omega_{h^{-1}\gamma h g^{-1}} \in \wedge F_{(y, yg^{-1})}^* \otimes L|_{(y, yg^{-1})}$ and $h_* \omega_{h^{-1}\gamma h g^{-1}} \in \wedge F_{(x, xg^{-1})}^* \otimes L|_{(x, xg^{-1})}$. Hence under the map i^g , it goes to $\wedge (F_1^g)_{(x, xg^{-1})}^* \otimes L|_{(x, xg^{-1})}$. Since $\text{Tr}_g \omega$ is H_1^g -invariant, it defines an element in $\Omega^\bullet(M^g, L^g)$.

Lemma 4.16. We have

$$\text{Tr}_g(\omega_1 * \omega_2) = (-1)^{|\omega_1||\omega_2|} \text{Tr}_g((g^{-1}\omega_2) * \omega_1)$$

PROOF. We compute

$$\begin{aligned} \mathrm{Tr}_g(\omega_1 * \omega_2)_\gamma &= i^g \int_{h \in H_1^x} \int_{h' \in H_1^{s(h)}} h_*((\omega_1)_{h'} \cdot (\omega_2)_{(h')^{-1}h^{-1}\gamma h^{g-1}}) \lambda^{s(h)}(dh') \lambda^x(dh) \\ &= i^g \int_{h, k \in H_1^x} h_*((\omega_1)_{h^{-1}k} \cdot (\omega_2)_{k^{-1}\gamma h^{g-1}}) \lambda^x(dh) \lambda^x(dk) \quad (k = hh') \\ \mathrm{Tr}_g((g^{-1}\omega_2) * \omega_1)_\gamma &= i^g \int_{h, k \in H_1^x} k_*((g^{-1}\omega_2)_{k^{-1}h} \cdot (\omega_1)_{h^{-1}\gamma k^{g-1}}) \lambda^x(dh) \lambda^x(dk). \end{aligned}$$

Replacing k by γk^{g-1} in the expression for $\mathrm{Tr}_g(\omega_1 * \omega_2)_\gamma$, we get

$$\mathrm{Tr}_g(\omega_1 * \omega_2)_\gamma = i^g \int_{h, k \in H_1^x} h_*((\omega_1)_{h^{-1}\gamma k^{g-1}} \cdot ((g^{-1}\omega_2)_{k^{-1}h})^{g^{-1}}) \lambda^x(dh) \lambda^x(dk).$$

It thus remains to prove that

$$i^g(h^{-1}k)_*(\alpha_2 \cdot \alpha_1) = (-1)^{|\alpha_1||\alpha_2|} i^g \alpha_1 \cdot \alpha_2^{g^{-1}}$$

for all $\alpha_1 \in (\wedge F^* \otimes L)|_{h^{-1}\gamma k^{g-1}}$ and $\alpha_2 \in (\wedge F^* \otimes L)|_{k^{-1}h}$. Replacing γ by $k^{-1}\gamma k^{g-1}$ and h by $k^{-1}h$ to simplify notations, it suffices to show:

$$i^g[(h^{-1})_*(\alpha_2 \cdot \alpha_1)] = (-1)^{|\alpha_1||\alpha_2|} i^g[\alpha_1 \cdot (\alpha_2^{g^{-1}})]$$

for all $\alpha_1 \in (\wedge F^* \otimes L)|_{h^{-1}\gamma}$ and $\alpha_2 \in (\wedge F^* \otimes L)|_h$. We may assume that $\alpha_1 = \eta_1 \otimes \xi_1$ and $\alpha_2 = \eta_2 \otimes \xi_2$, for $\eta_1, \eta_2 \in \wedge F^*$ and $\xi_1, \xi_2 \in L$. It then suffices to establish the following equalities:

- (a) $(h^{-1})_*(\xi_2 \cdot \xi_1) = \xi_1 \cdot \xi_2^{g^{-1}};$
- (b) $i^g(h^{-1})_*[(r_{h^{-1}\gamma}\eta_2) \wedge (l_h\eta_1)] = i^g(-1)^{|\eta_1||\eta_2|}(r_{h^{g-1}}\eta_1) \wedge (l_{h^{-1}\gamma}\eta_2^{g^{-1}}).$

For (a), choose a lift $\tilde{h} \in \widetilde{H}_1$ of h , and identify \tilde{h} to $(\tilde{h}, 1) \in \widetilde{H}_1 \times_{S^1} \mathbb{C} = L$. Then, from (??), $(h^{-1})_*(\xi_2 \cdot \xi_1) = (\tilde{h})^{-1}(\xi_2 \cdot \xi_1)(\tilde{h}^{g^{-1}}) = (\tilde{h}^{-1}\xi_2)(\xi_1\tilde{h}^{g^{-1}})$. Now, $\tilde{h}^{-1}\xi_2$ is an element of $L_{s(h)} \cong \mathbb{C}$, hence can be identified to the complex number $(\tilde{h}^{-1}\xi_2)^{g^{-1}} \in L_{s(h)g^{-1}} \cong \mathbb{C}$. Therefore we have $(h^{-1})_*(\xi_2 \cdot \xi_1) = (\xi_1\tilde{h}^{g^{-1}})(\tilde{h}^{-1}\xi_2)^{g^{-1}} = \xi_1\xi_2^{g^{-1}}$ as claimed.

For (b), using the fact that $(h^{-1})_* = l_{h^{-1}}r_{h^{g-1}}$, we see that (b) reduces to

$$i^g l_{h^{-1}} r_{h^{-1}\gamma h^{g-1}} \eta_2 = i^g l_{h^{-1}\gamma} \eta_2^{g^{-1}}.$$

We can of course assume that $\eta_2 \in F_h^*$. By duality, this is equivalent to

$$l_{h^{-1}} r_{h^{-1}\gamma h^{g-1}} X = l_{h^{-1}\gamma} X^{g^{-1}},$$

i.e. to $r_{h^{-1}\gamma h^{g-1}} X = l_\gamma X^{g^{-1}}$ for all $X \in (F_1)_h^g$. Since $(\sigma \circ s)_* : (F_1)_{\gamma h^{g-1}} \rightarrow T_{\sigma \circ s(h)} M^g$ is an isomorphism, it suffices to check that both sides coincide after applying the map $(\sigma \circ s)_*$.

$$(\sigma \circ s)_*(l_\gamma X^{g^{-1}}) = (\sigma \circ s)_*(X^{g^{-1}}) = ((\sigma \circ s)_*(X))^{g^{-1}} = (\sigma \circ s)_*(X)$$

since $(\sigma \circ s)_*(X) \in M^g$ by assumption. On the other hand,

$$(\sigma \circ s)_*(r_{h^{-1}\gamma h^{g-1}} X) = (\sigma \circ t)_*(r_{h^{-1}\gamma h^{g-1}} X) = (\sigma \circ t)_*(X) = (\sigma \circ s)_*(X).$$

This completes the proof. \square

Lemma 4.17. *We have*

$$Tr_g \circ \nabla = \nabla_g \circ Tr_g.$$

PROOF. Since the map Tr_g factors through the restriction to M^g , we may simply assume that $M = M^g$, and thus $H_0 = H_0^g$.

For all $y \in H_0$, the element $\omega_{(y, yg^{-1})} \in \Lambda F_{(y, yg^{-1})}^* \otimes L_{(y, yg^{-1})}$ restricts to an element $\omega'_y \in \Lambda F_y^* \otimes L_{(y, yg^{-1})}$.

For all $x \in H_0$, we have

$$(\text{Tr}_g \omega)(X_1, \dots, X_n)(\sigma(x)) = \int_{h=(x, y) \in H_1^x} h_* \omega'(\tilde{X}_1(y), \dots, \tilde{X}_n(y)) \lambda^x(dh).$$

Then

$$\begin{aligned} & (\nabla_g(\text{Tr}_g \omega))(X_1, \dots, X_n)(x) \\ &= \sum_{i=1}^n (-1)^{i+1} \nabla_{X_i}^g \cdot (\text{Tr}_g \omega)(X_1, \dots, \widehat{X_i}, \dots, X_n) \\ & \quad + \sum_{1 \leq i < j \leq n} (-1)^{i+j} (\text{Tr}_g \omega)([X_i, X_j], X_1, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_n) \\ &= \sum_{i=1}^n (-1)^{i+1} \nabla_{X_i}^g \cdot h_* \int_{h \in H_1^x} \omega'(\tilde{X}_1(y), \dots, \widehat{\tilde{X_i}(y)}, \dots, \tilde{X}_n(y)) \lambda^x(dh) \\ & \quad + \sum_{1 \leq i < j \leq n} (-1)^{i+j} \int_{h \in H_1^x} h_* \omega'([\tilde{X_i}, \tilde{X_j}](y), \tilde{X}_1(y), \dots, \widehat{\tilde{X_i}(y)}, \dots, \widehat{\tilde{X_j}(y)}, \dots, \tilde{X}_n(y)) \lambda^x(dh) \\ &= \sum_{i=1}^n (-1)^{i+1} \nabla_{X_i}^g \cdot \int_{h \in H_1^x} h_* \omega'(\tilde{X}_1(y), \dots, \widehat{\tilde{X_i}(y)}, \dots, \tilde{X}_n(y)) \lambda^x(dh) \\ & \quad + \sum_{1 \leq i < j \leq n} (-1)^{i+j} \int_{h \in H_1^x} h_* \omega'([\tilde{X_i}, \tilde{X_j}](y), \tilde{X}_1(y), \dots, \widehat{\tilde{X_i}(y)}, \dots, \widehat{\tilde{X_j}(y)}, \dots, \tilde{X}_n(y)) \lambda^x(dh). \end{aligned}$$

The last equality is a consequence of integrability of the bundle F' .

To conclude, we need to show that in the expression $\nabla_{X_i}^g \cdot \int_{h \in H_1^x} \omega'(\tilde{X}_1(y), \dots, \widehat{\tilde{X_i}(y)}, \dots, \tilde{X}_n(y)) \lambda^x(dh)$, derivation commutes with integration, i.e. that for every L -valued section the equality

$$\nabla_{\tilde{X}} \int_{h=(x, y) \in H_1^x} h_* \xi(y) \lambda^x(dh) = \int_{h=(x, y) \in H_1^x} h_* ((\nabla_{\tilde{X}} \xi)(y)) \lambda^x(dh)$$

holds; this is a consequence of Lemma 4.5. \square

The following results can be easily verified directly.

Proposition 4.18. *The family of trace maps $\text{Tr}_g : \Omega^\bullet \rightarrow \bar{\Omega}_g$ is G -equivariant, i.e., the following diagram*

$$\begin{array}{ccc} \Omega^\bullet & \xrightarrow{\text{Tr}_g} & \bar{\Omega}_g := \Omega_c^\bullet(M^g, L^g) \\ & \searrow \text{Tr}_{h^{-1}gh} & \downarrow (\phi_h^{-1})^* \\ & & \bar{\Omega}_{h^{-1}gh} := \Omega_c^\bullet(M^h, L^h) \end{array}$$

commutes.

5. DISCUSSIONS AND OPEN QUESTIONS

5.1. Global equivariant differential forms a la Block-Getzler. In this section, we briefly recall the basic construction of global equivariant differential forms a la Block-Getzler. We closely follow the approach of [8].

Recall that G acts on the manifold underlying G by conjugation: $h \cdot g = g^{-1}hg$. Equip G with the topology of invariant open sets:

$$(23) \quad \mathcal{O} = \{U \subset G \text{ open} | U = U \cdot g \ \forall g \in G\}.$$

Construct a sheaf $\bar{\Omega}^\bullet(M, G)$ over G as follows. The stalk of the sheaf $\bar{\Omega}^\bullet(M, G)$ at $g \in G$ is the space of equivariant differential forms $\bar{\Omega}^\bullet(M, G)_g = \bar{\Omega}_{G^g}(M^g)$, which consist of germs at zero of smooth maps from \mathfrak{g}^g to $\Omega(M^g)$ equivariant under G^g . It is easy to see that if $\omega \in \bar{\Omega}^\bullet(M, G)_g$, $k \cdot \omega \in \bar{\Omega}^\bullet(M, G)_{g \cdot k}$. Therefore the group G acts on the sheaf $\bar{\Omega}^\bullet(M, G)$ in a way compatible with its conjugation action on G . Moreover, the equivariant differential coboundary operators on $\{\bar{\Omega}_{G^g}^\bullet(M^g), g \in G\}$ are compatible with the G action as well. That is

$$k \cdot d_{G^g} \omega = d_{G^{Ad_k g}}(k \cdot \omega),$$

where $d_{G^g} : \bar{\Omega}^\bullet(M, G)_g \rightarrow \bar{\Omega}^\bullet(M, G)_g$ is the equivariant differential on $\bar{\Omega}^\bullet(M, G)_g$.

Definition 5.1. We say that a point $h = g \exp X \in G^g$, $X \in \mathfrak{g}^g$, is near the point $g \in G^g$ if $M^h \subseteq M^g$ and $G^h \subseteq G^g$.

Note that, from a theorem of Mostow-Palais [20, 22], it follows that the set of all points in G^g near g is indeed an open neighborhood of g in G under the topology given as in (23). Hence a section $\omega \in \Gamma(U, \bar{\Omega}^\bullet(M, G))$ of the sheaf $\bar{\Omega}^\bullet(M, G)$ over an invariant open set $U \subset G$ is given by, for each point $g \in U$, an element $\omega_g \in \bar{\Omega}^\bullet(M, G)_g$, such that if $h = g \exp X \in G^g$ is near g , we have the equality of germs:

$$\omega_h(Y) = \omega_g(X + Y) \in \bar{\Omega}^\bullet(M, G)_h, \quad \forall Y \in \mathfrak{g}^h.$$

Thus $\bar{\Omega}^\bullet(M, G)$ is an equivariant sheaf of differential graded algebras over G . By a global equivariant differential form on G , we mean an equivariant section $\omega \in \Gamma(G, \bar{\Omega}^\bullet(M, G))^G$, i.e. $\omega_{k \cdot g} = k \cdot \omega_g$, $\forall g, k \in G$.

Remark 5.2. To understand the meaning of the above conditions on global equivariant differential forms, it is useful to consider the following simple example in an analogous situation. Let $f \in C^\infty(G)^G$. For each fixed $g \in G$, denote $f_g(X)$ the germ of the function $X \rightarrow f(g \exp X)$, $\forall X \in \mathfrak{g}^g$. It is easy to check that the following identities hold

- (1) if $h = g \exp X \in G^g$, $X \in \mathfrak{g}^g$, then $f_h(Y) = f_g(X + Y)$, $\forall Y \in \mathfrak{g}^g$;
- (2) for any $r \in G$, $f_{r^{-1}gr} = (Ad_{r^{-1}})^* f_g$.

Let

$$\mathcal{A}_G^\bullet(M) = \Gamma(G, \bar{\Omega}^\bullet(M, G))^G$$

be the space of global equivariant differential forms on G . The family of differentials $\{d_{G^g}\}_{g \in G}$ induces a differential d_{eq} on $\mathcal{A}_G^\bullet(M)$. The delocalized equivariant cohomology $H_{G, delocalized}^\bullet(M)$ is defined as its $\mathbb{Z}/2$ -graded cohomology:

$$H_{G, delocalized}^\bullet(M) := H^\bullet(\mathcal{A}_G^\bullet(M), d_{eq}).$$

The following result is due to Block-Getzler [8] when the Lie group G is compact, and to Baum-Brylinski-MacPherson [3] when $G = S^1$.

Theorem 5.3. *Let G be a compact Lie group, M a compact manifold on which G acts smoothly. Then*

$$HP_\bullet^G(C^\infty(M)) \cong H^\bullet(\mathcal{A}_G^\bullet(M), d_{eq}).$$

5.2. Delocalized twisted equivariant cohomology. A natural open question arises:

Question 5.4. *Introduce global twisted equivariant differential forms by modifying the notion of global equivariant differential forms of Block-Getzler [8] in the previous section, and define delocalized twisted equivariant cohomology.*

For any $\alpha \in H_G^3(M, \mathbb{Z})$, let $\coprod_{g \in G} P^g$ be a family of G -equivariant flat S^1 -bundles as in Theorem 2.14, and $L = \coprod_{g \in G} L^g$, where $L^g = P^g \times_{S^1} \mathbb{C}$, $\forall g \in G$, are their associated G -equivariant flat complex line bundles.

Following Block-Getzler [8], the key issue is to introduce a sheaf $\bar{\Omega}^\bullet(M, G, L)$ over G whose stalk at $g \in G$ is the space of equivariant differential forms, with coefficients in L^g : $\bar{\Omega}^\bullet(M, G, L)_g = \bar{\Omega}_{G^g}(M^g, L^g)$.

For this purpose, we propose the following

Conjecture 5.5. *Assume that a point $h = g \exp X \in G^g$, $X \in \mathfrak{g}^g$, is near the point $g \in G^g$. Then there exists a canonical isomorphism of flat S^1 -bundles over M^h :*

$$(24) \quad \phi_{gh} : P^h \rightarrow P^g|_{M^h},$$

where $P^g|_{M^h} \rightarrow M^h$ denotes the restriction of $P^g \rightarrow M^g$ under the inclusion $M^h \subseteq M^g$.

If the conjecture above holds, one can make sense of a *global twisted equivariant differential form* on G by defining it to be an equivariant section $\omega \in \Gamma(G, \bar{\Omega}^\bullet(M, G, L))^G$, i.e. a family $\{\omega_g \in \bar{\Omega}_{G^g}(M^g, L^g) | g \in G\}$ such that

- (1) $\omega_{k \cdot g} = k \cdot \omega_g$, $\forall g, k \in G$, and
- (2) if $h = g \exp X \in G^g$, $X \in \mathfrak{g}^g$ is near g , we have the equality of germs

$$\phi_{gh}[\omega_h(Y)] = \omega_g(X + Y)|_{M^h} \in \bar{\Omega}^\bullet(M, G, L)_h, \quad \forall Y \in \mathfrak{g}^h,$$

where $\phi_{gh} : L^h \rightarrow L^g$ is the canonical isomorphism induced by the isomorphism of S^1 -bundles as in Eq. (24).

Let $\mathcal{A}_G^\bullet(M, L) = \Gamma(G, \bar{\Omega}^\bullet(M, G, L))$ be the space of global twisted equivariant differential forms on G . The family of differentials $\{d_{G^g}^\alpha\}_{g \in G}$ induces a differential d_{eq}^α on $\mathcal{A}_G^\bullet(M, L)$. By abuse of notations, we write

$$d_{\text{eq}}^\alpha = \nabla + \iota - 2\pi i \eta_G.$$

The delocalized twisted equivariant cohomology $H_{G, \text{delocalized}, \alpha}^\bullet(M)$ can then be defined as its $\mathbb{Z}/2$ -graded cohomology:

$$H_{G, \text{delocalized}, \alpha}^\bullet(M) := H^\bullet(\mathcal{A}_G^\bullet(M, L), d_{\text{eq}}^\alpha).$$

Remark 5.6. A possible way to construct ϕ_{gh} in Eq. (24) is to use the parallel transportation along the path $t \mapsto g \exp(tX)$. More precisely, assume that $S^1 \rightarrow \tilde{\Gamma} \rightarrow \Gamma \rightrightarrows M'$ is an S^1 -central extension representing $\alpha \in H_G^3(M, \mathbb{Z})$, where $f : M' \rightarrow M$ is a surjective submersion. Let $\nabla \in \Omega^1(\tilde{\Gamma})$ be a gerbe connection. The fibers P_x^g and P_x^h can be identified with $P_{x, g, x}$ and $P_{x, h, x}$, respectively. Then parallel transportation along the path $t \mapsto (x, g \exp(tX), x)$ can thus be used to identify $P_{x, g, x} := \tilde{\Gamma}|_{(x, g, x)}$ with $P_{x, h, x} := \tilde{\Gamma}|_{(x, h, x)}$. However, for this identification to intertwine the connections ∇^g and ∇^h , one needs that the curvature 2-form $\omega \in \Omega^2(\Gamma)$ of the connection ∇ satisfies the condition

$$(25) \quad \omega_{x, g, x}((v, 0, v), (0, X, 0)) = 0,$$

$\forall x \in M^h \subset M^g$ and all $v \in T_x M^h$. This condition does not necessarily always hold as indicated by the example below.

Let M be S^1 endowed with a trivial action of $G = S^1$. Let $\Gamma \rightrightarrows M$ be the transformation groupoid $M \rtimes G \rightrightarrows M$. Consider

$$\tilde{\Gamma} = \frac{[0, 1] \times G \times S^1}{(0, g, \lambda) \sim (1, g, g + \lambda)}.$$

(The product in S^1 is written additively.) It is clear that $M \times S^1 \rightarrow \tilde{\Gamma} \rightarrow \Gamma \rightrightarrows M$ is an S^1 -central extension, where the map $\tilde{\Gamma} \rightarrow \Gamma$ is defined as $(u, g, \lambda) \mapsto (u, g)$ (with S^1 identified to \mathbb{R}/\mathbb{Z}), and the map $M \times S^1 \rightarrow \tilde{\Gamma}$ is $(u, \lambda) \mapsto (u, 0, \lambda)$. Since $\tilde{\Gamma} \rightarrow \Gamma$ is a non-trivial S^1 -principal bundle, its first Chern class must be nonzero. Therefore, Condition (25) fails in this case.

5.3. De Rham model of equivariant twisted K-theory. Let A be a topological associative algebra, and G a compact Lie group acting on A by automorphisms. Then there is an equivariant Chern character [7, 10, 11]:

$$(26) \quad \text{ch}^G : K_{\bullet}^G(A) \rightarrow HP_{\bullet}^G(A)$$

from the equivariant K -theory of A to the periodic cyclic homology $HP_{\bullet}^G(A)$.

Let $R(G)$ be the representation ring of G , and $R^{\infty}(G)$ the algebra $C^{\infty}(G)^G$ of smooth functions on the group G invariant under the conjugation. Since the character map sends $R(G)$ to $R^{\infty}(G)$, $R^{\infty}(G)$ is an algebra over the ring $R(G)$. The following result is due to Block [7] and Brylinski [10, 11].

Theorem 5.7. *Let G be a compact Lie group and A a topological G -algebra. Then the equivariant Chern character (26) induces an isomorphism*

$$(27) \quad HP_{\bullet}^G(A) \cong K_{\bullet}^G(A) \otimes_{R(G)} R^{\infty}(G)$$

Apply the theorem above to the topological algebra:

$$A = C_c^{\infty}(H, L).$$

By definition (see [25] for details), $K_{\bullet}^G(C_c^{\infty}(H, L))$ is exactly the twisted K -theory group $K_{G, \alpha}^{\bullet}(M)$. Thus we obtain

Corollary 5.8. *Under the same hypothesis as in Theorem 4.8, we have*

$$(28) \quad HP_{\bullet}^G(C_c^{\infty}(H, L)) \cong K_{G, \alpha}^{\bullet}(M) \otimes_{R(G)} R^{\infty}(G)$$

Therefore one may think of $HP_{\bullet}^G(C_c^{\infty}(H, L))$ as a de Rham model of equivariant twisted K -theory.

Conjecture 5.9. *The family of chain maps $\{\tau_g\}_{g \in G}$ as in Theorem 4.8 induces a quasi-isomorphism*

$$(29) \quad (PC_{\bullet}^G(C_c^{\infty}(H, L)), b + \mathcal{B}) \rightarrow (\mathcal{A}_G^{\bullet}(M, L), d_{eq}^{\alpha})$$

An immediate consequence yields the following

Conjecture 5.10. *Let G be a compact Lie group, M a compact manifold on which G acts smoothly. For any $\alpha \in H_G^3(M, \mathbb{Z})$, the equivariant Chern character composing with the isomorphism induced by (29) leads to a natural isomorphism*

$$(30) \quad K_{G, \alpha}^{\bullet}(M) \otimes_{R(G)} R^{\infty}(G) \xrightarrow{\sim} H_{G, \text{delocalized}, \alpha}^{\bullet}(M)$$

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